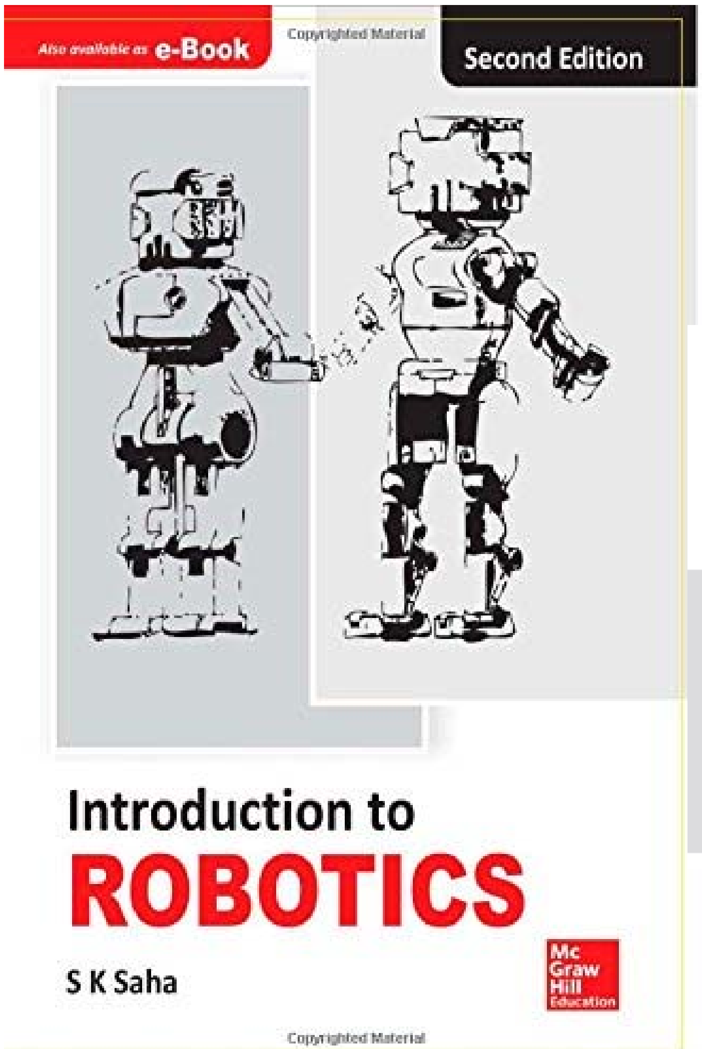


Lecture 3
Robot Kinematics
by
S.K. Saha
Aug. 22'16 (M)@JRL301 (Robotics Tech.)

5 Transformations

6 Kinematics

Recap

- Kinematic chain: Links and joints
- DOF: Parameters-constraints
- Position: Simple (like good friend in the hostel)
- Orientation: Confusing and SERIOUS attention to be paid

Denavit and Hartenberg (DH) Parameters—Frame Allotment

- Serial chain

- Two links connected by revolute joint, or
- Two links connected by prismatic joint

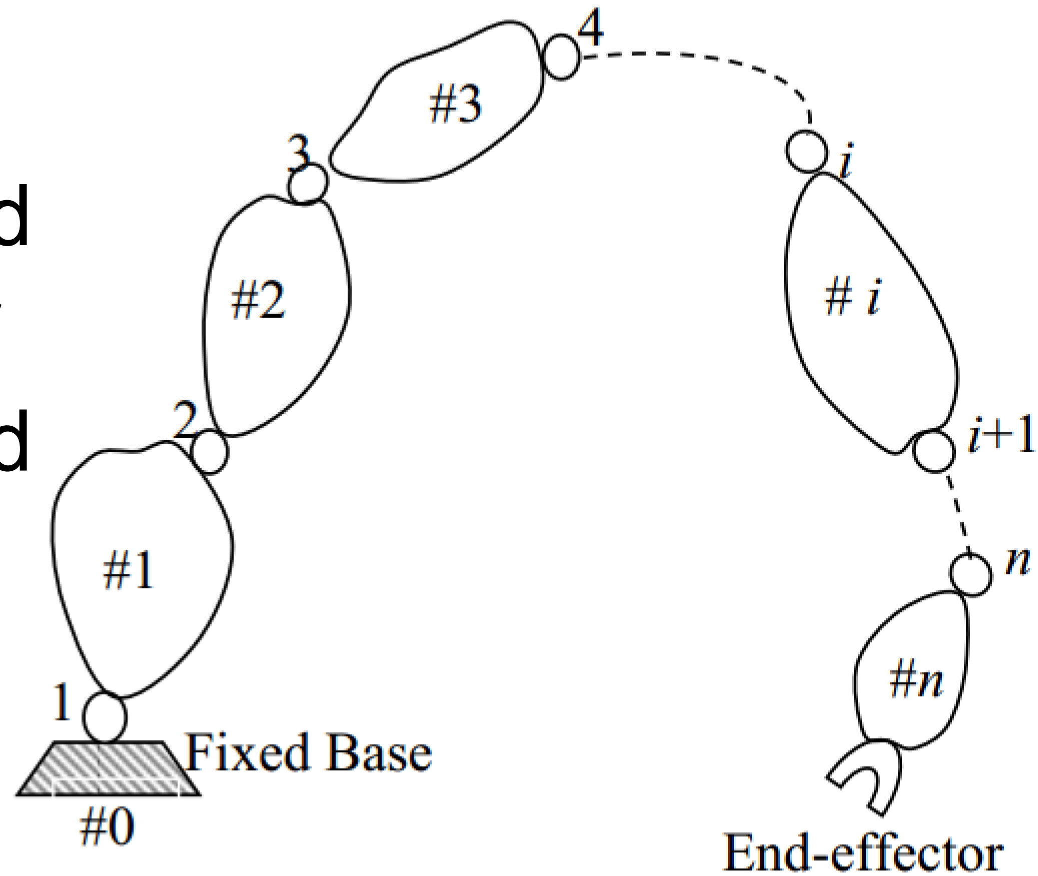
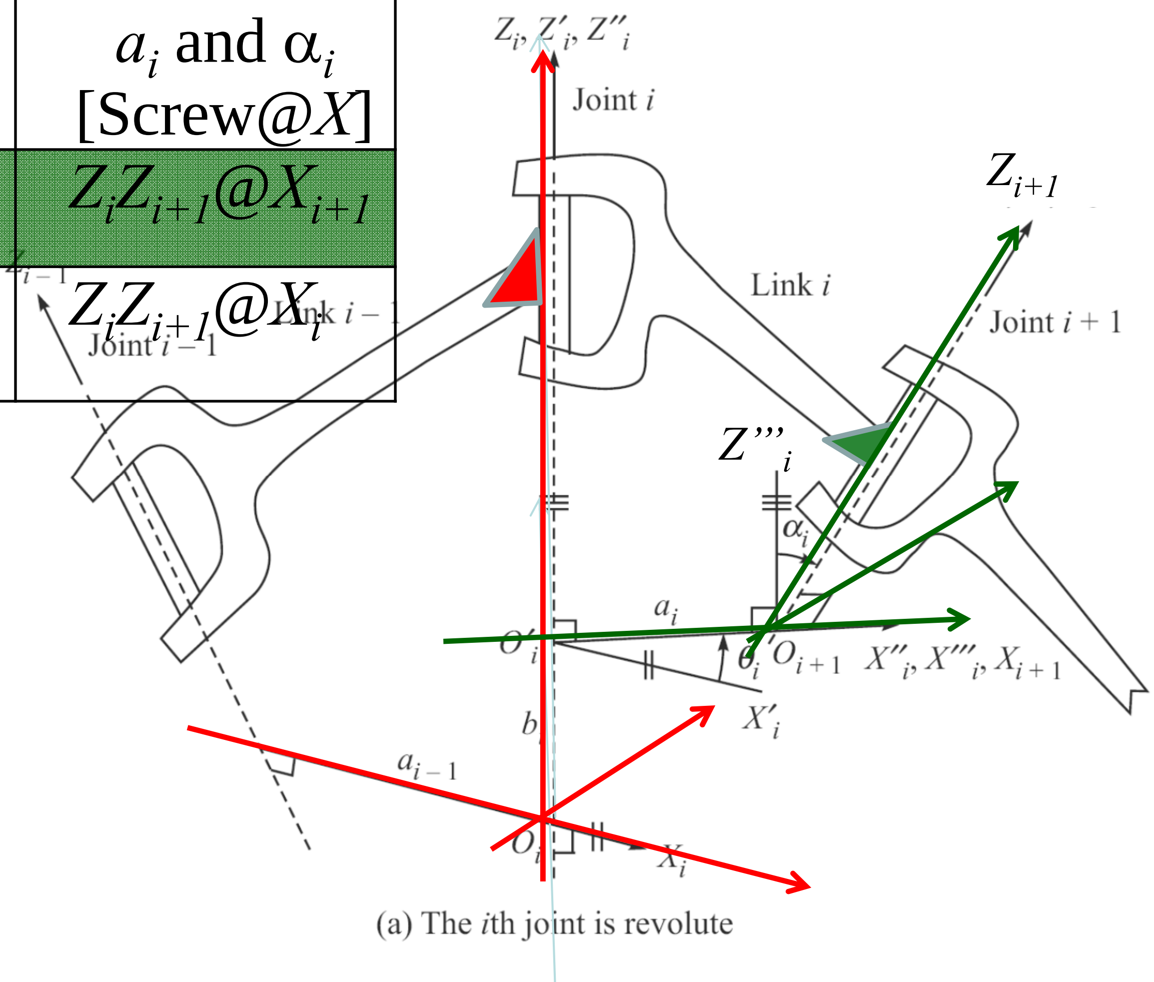


Fig. 5.27 Serial manipulator

- Joint axis *i*: Link *i-1* + link *i*
- Link *i*: Fixed to **frame *i+1*** (Saha) / **frame *i*** (Craig)

DH	Variables b_i and θ_i [Screw@Z]	Constants a_i and α_i [Screw@X]
Saha	$X_iX_{i+1}@Z_i$	$Z_iZ_{i+1}@X_{i+1}$
Craig	$X_{i-1}X_i@Z_i$	$Z_iZ_{i+1}@X_i$



(a) The *i*th joint is revolute

Revolute Joint

- DH@Z (Variable)
  - Joint offset (b)
  - Joint angle (θ)

- DH@X (Const.)
  - Link length (a)
  - Twist angle (α)

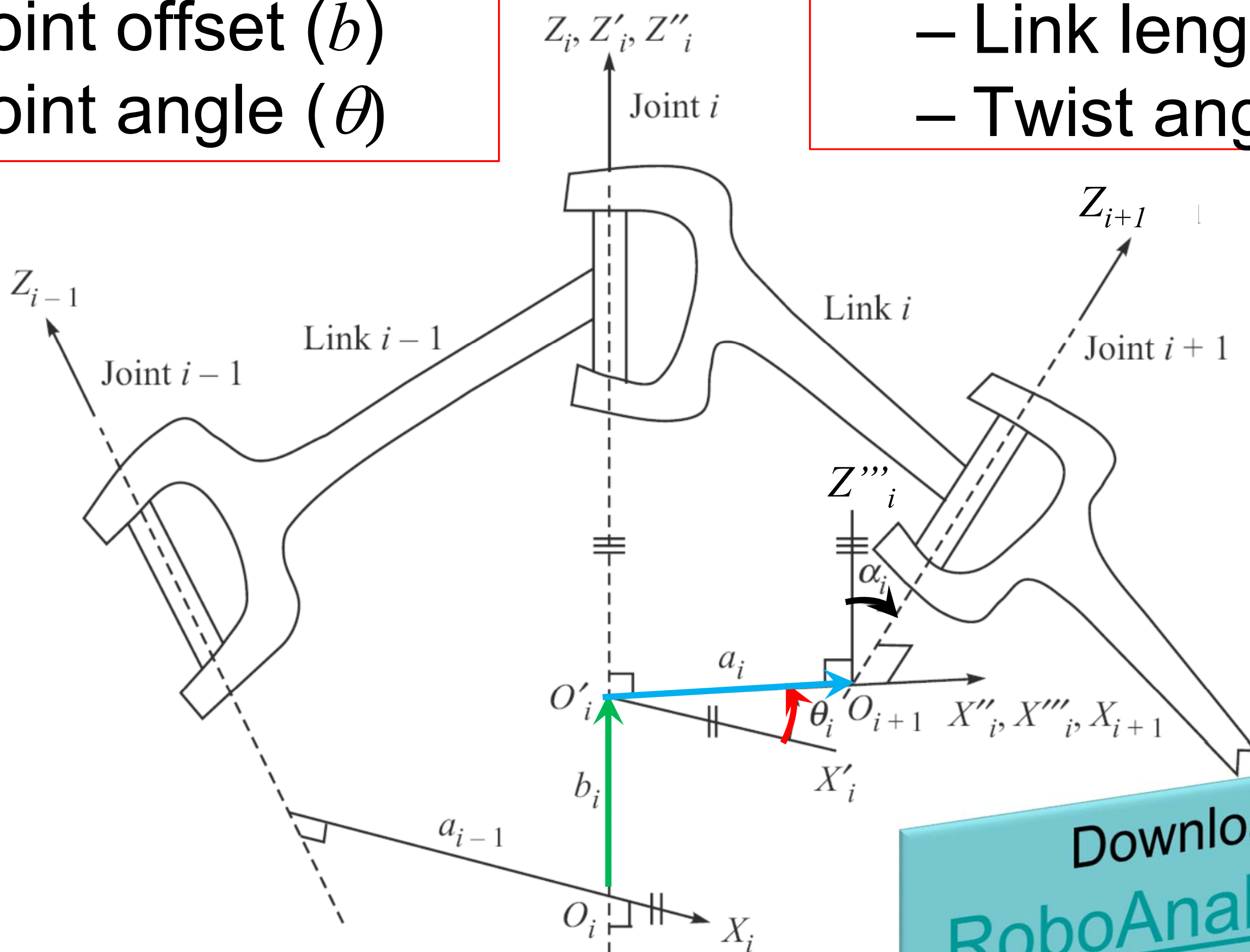


Fig. 5.28 (a) The *i*th joint is revolute

Download RoboAnalyzer?

Mathematically

- Translation along Z_i

$$\mathbf{T}_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \ldots (5.49a)$$

- Rotation about Z_i

$$\mathbf{T}_\theta = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \ldots (5.49b)$$

- Translation along X_{i+1}

$$\mathbf{T}_a = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \ldots (5.49c)$$

- Rotation about X_{i+1}

$$\mathbf{T}_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_i & -S\alpha_i & 0 \\ 0 & S\alpha_i & C\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \ldots (5.49d)$$

- Total transformation from Frame *i* to Frame *i+1*

$$\mathbf{T}_i = \mathbf{T}_b \mathbf{T}_\theta \mathbf{T}_a \mathbf{T}_\alpha \qquad \ldots (5.50a)$$

$$\mathbf{T}_i = \begin{bmatrix} \multicolumn{3}{c}{\text{Rotation Matrix}} & \text{Position} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\ldots (5.50b)$$

Three-link Planar Arm

- DH-parameters

Link	b_i	θ_i	a_i	α_i
1	Fill-up the DH parameters			
2				
3				

- Frame transformations (Homogeneous)

$$\mathbf{T}_i = \begin{bmatrix} \text{Fill-up with the elements} & & & \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ for } i=1,2,3$$

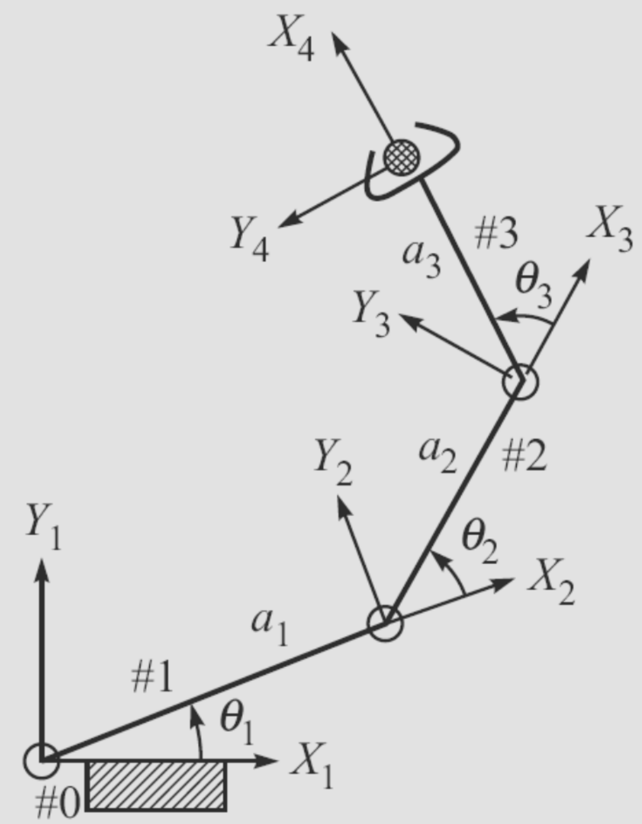


Fig. 5.24 A three-link planar arm

Forward and Inverse Kinematics

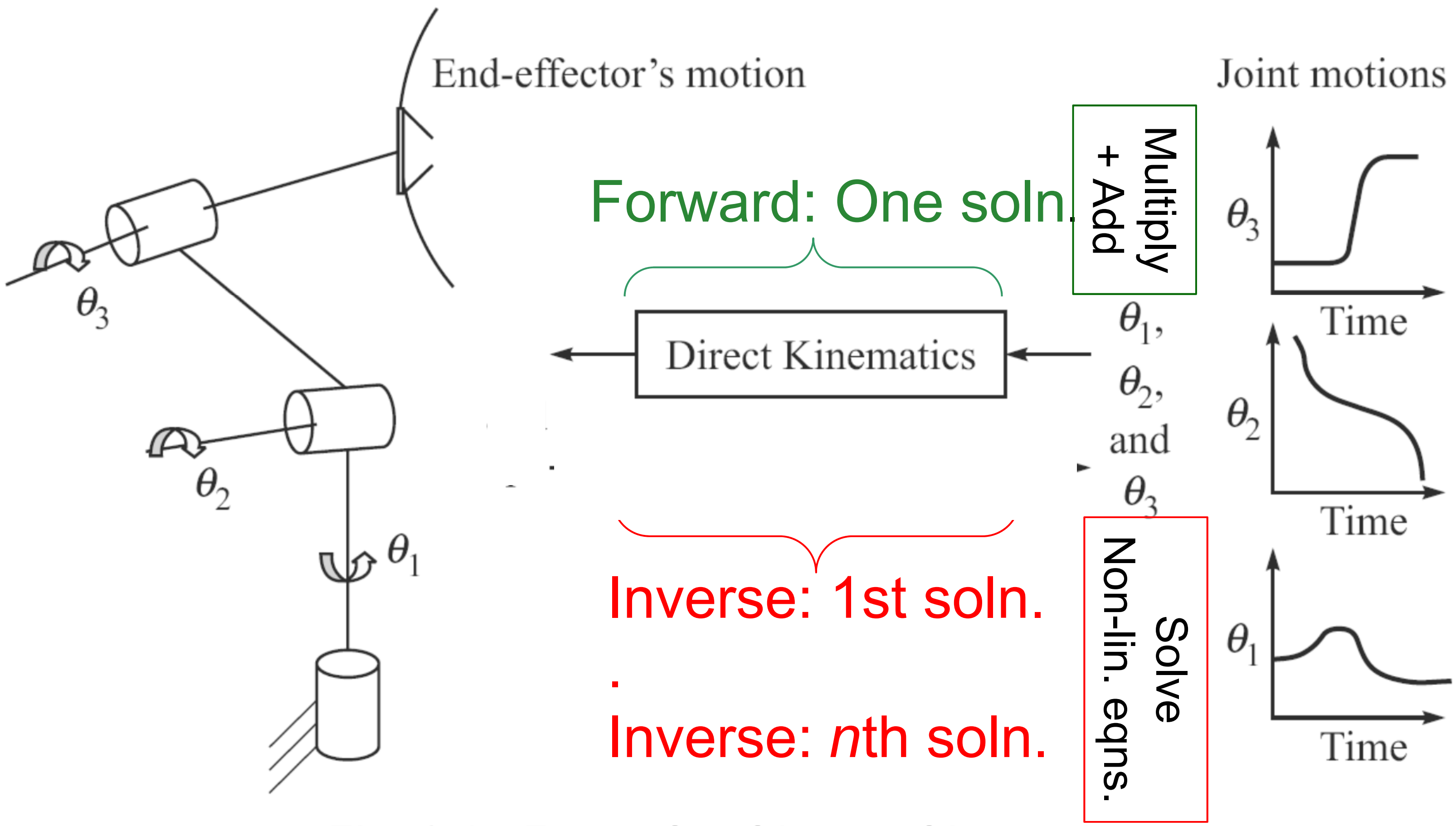


Fig. 6.1 Forward and inverse kinematics

Overall Transformation for Three-link Arm

$$\mathbf{T} = \mathbf{T}_1\, \mathbf{T}_2\, \mathbf{T}_3$$

$$\mathbf{T} = \begin{bmatrix} C\theta_{123} & -S\theta_{123} & 0 & a_1 C\theta_1 + a_2 C\theta_{12} + a_3 C\theta_{123} \\ S\theta_{123} & C\theta_{123} & 0 & a_1 S\theta_1 + a_2 S\theta_{12} + a_3 S\theta_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DH Parameters of Articulated Arm

Link	b_i	θ_i	a_i	α_i
1	0	θ_1 (JV)	0	$-\pi/2$
2	0	θ_2 (JV)	a_2	0
3	0	θ_3 (JV)	a_3	0

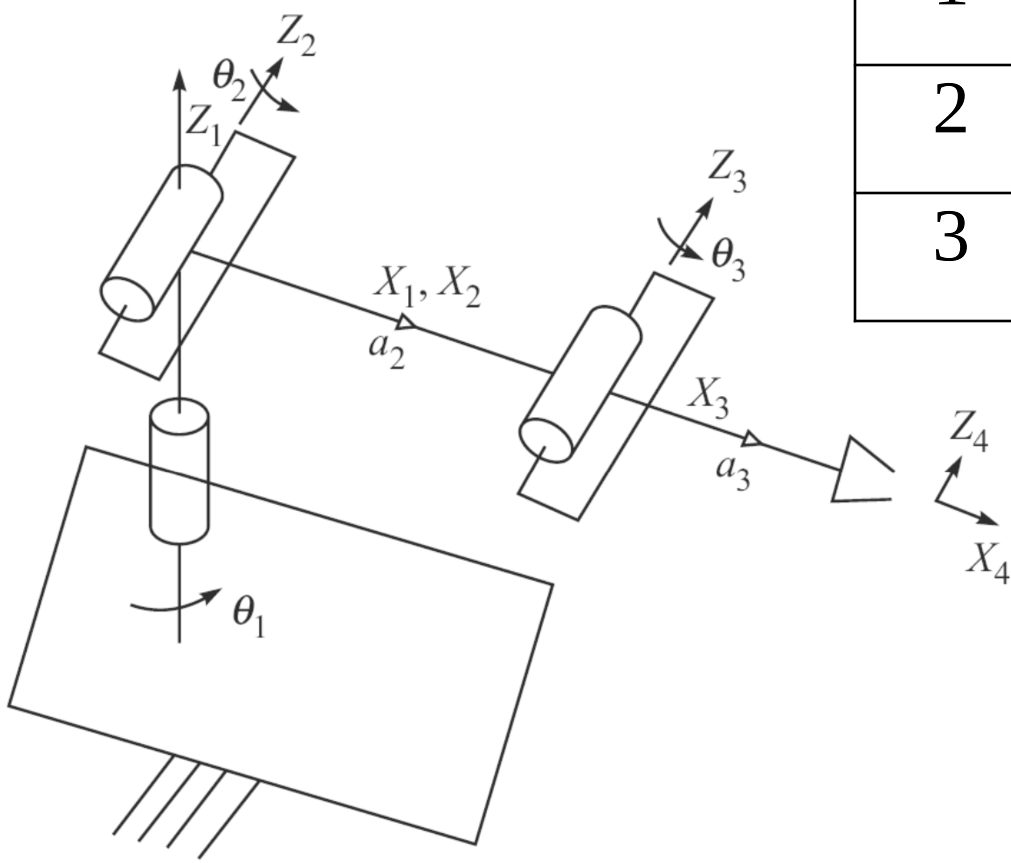


Fig. 5.29 An articulated arm

Matrices for Articulated Arm

$$\mathbf{T}_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{T}_2 \equiv \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_3 \equiv \begin{bmatrix} c_3 & -s_3 & 0 & a_3 c_3 \\ s_3 & c_3 & 0 & a_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

MATLAB

$$\mathbf{T} \equiv \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & -s_1 & c_1 (a_2 c_2 + a_3 c_{23}) \\ s_1 c_{23} & -s_1 s_{23} & c_1 & s_1 (a_2 c_2 + a_3 c_{23}) \\ -s_{23} & -c_{23} & 0 & -(a_2 s_2 + a_3 s_{23}) \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \ldots (6.11)$$

Inverse Kinematics

- Unlike Forward Kinematics, general solutions are not possible.
- Several architectures are to be solved differently.

Two-link Arm

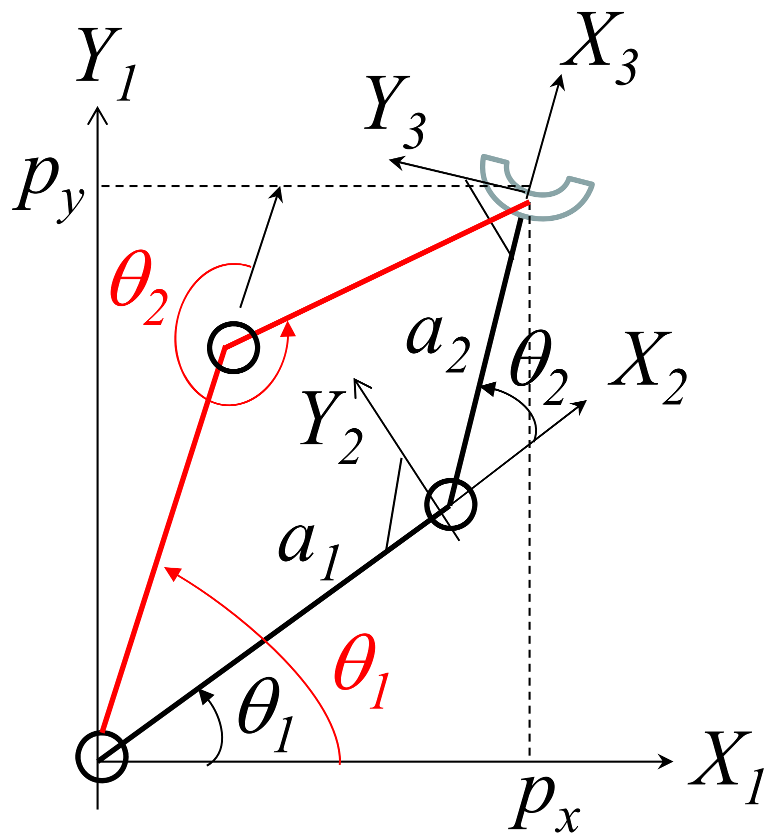
$$p_x = a_1 c_1 + a_2\, c_{12}$$

$$p_y = a_1 s_1 + a_2\, s_{12}$$

$$c_2 = \frac{p_x^2 + p_y^2 - a_1^2 - a_2^2}{2\, a_1\, a_2}$$

$$s_2 = \pm\sqrt{1 - c_2^2}$$

$$\theta_2 = \text{atan2}\,(s_2, c_2)$$



RoboAnalyzer

$$s_1 = \frac{(a_1 + a_2 c_2)\, p_y - a_2 s_2\, p_x}{\Delta}$$

$$\Delta \equiv a_1^2 + a_2^2 + 2 a_1 a_2 c_2 = p_x^2 + p_y^2$$

$$c_1 = \frac{(a_1 + a_2 c_2)\, p_x + a_2 s_2\, p_y}{\Delta}$$

$$\theta_1 = \text{atan2}\,(s_1, c_1)$$

Inverse Kinematics of 3-DOF RRR Arm

$$\varphi = \theta_1 + \theta_2 + \theta_3 \quad \ldots (6.18a)$$

$$p_x = a_1 c_1 + a_2 c_{12} + a_3 c_{123} \quad \ldots (6.18b)$$

$$p_y = a_1 s_1 + a_2 s_{12} + a_3 s_{123} \quad \ldots (6.18c)$$

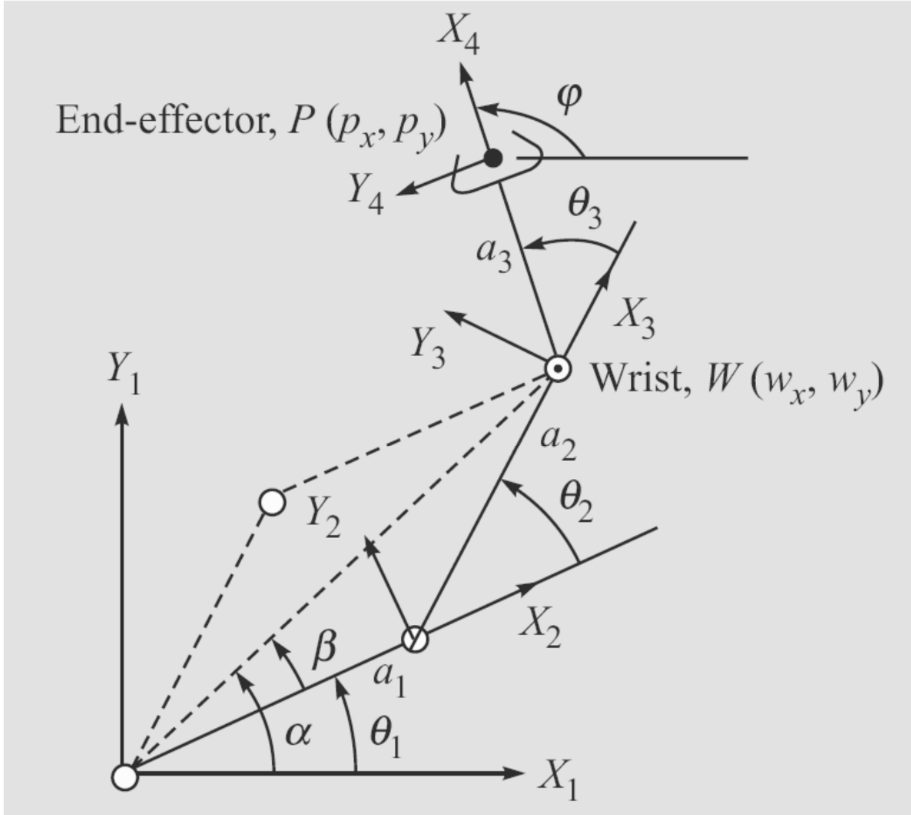


Fig. 6.3 Kinematics of a three-link planar arm

$$w_x = p_x - a_3 c\,\varphi = a_1 c_1 + a_2 c_{12} \quad \ldots (6.19a)$$

$$w_y = p_y - a_3 s\,\varphi = a_1 s_1 + a_2 s_{12} \quad \ldots (6.19b)$$

$$w^2_x + w^2_y = a_1^2 + a_2^2 + 2\, a_1 a_2 c_2 \qquad \ldots (6.20a)$$

$$c_2 = \frac{w_1^2 + w_2^2 - a_1^2 - a_2^2}{2\, a_1\, a_2} \quad s_2 = \pm\sqrt{1 - c_2^2} \qquad \ldots (6.20b,c)$$

$$\theta_2 = \text{atan2}\,(s_2, c_2) \qquad \ldots (6.21)$$

$$w_x = (a_1 + a_2 c_2) c_1 - a_2 s_1 s_2 \qquad \ldots (6.22a)$$

$$w_y = (a_1 + a_2 c_2) s_1 + a_2 c_1 s_2 \qquad \ldots (6.22b)$$

$$s_1 = \frac{(a_1 + a_2 c_2) w_y - a_2 s_2 w_x}{\Delta} \quad c_1 = \frac{(a_1 + a_2 c_2) w_x + a_2 s_2 w_y}{\Delta}$$

$$\Delta \equiv a_1^2 + a_2^2 + 2 a_1 a_2 c_2 = w_x^2 + w_y^2 \qquad \ldots (6.23a,b)$$

$$\theta_1 = \text{atan2}\,(s_1, c_1) \qquad \ldots (6.23c)$$

$$\theta_3 = \varphi - \theta_1 - \theta_2 \qquad \ldots (6.24)$$

Numerical Example

- An RRR planar arm (Example 6.15). Input

$\mathbf{T} \equiv$

Rotation Matrix

Origin of end-effector frame

4.23

1.86

0

$$\begin{matrix} 0 & 0 & 0 & 1 \end{matrix}$$

where $\phi = 60^o$, and $a_1 = a_2 = 2$ units, and $a_3 = 1$ unit.

Do it yourself → Verify using RoboAnalyzer

Using eqs. (6.13b-c), $c_2 = 0.866$, and $s_2 = 0.5$,

$\theta_2 = 30^o$

Next, from eqs. (6.16a-b), $s_1 = 0$, and $c_1 = 0.866$.

$\theta_1 = 0^o$.

Finally, from eq. (6.17) , $\theta_3 = 30^o$.

MATLAB program

Therefore $\theta_1 = 0^o\ \theta_2 = 30^o$, and $\theta_3 = 30$ …(6.30b)

The positive values of s_2 was used in evaluating $\theta_2 = 30^o$.

The use of negative value would result in :

$\theta_1 = 30^o\ \theta_2 = -30^o$, and $\theta_3 = 60^o$ …(6.30c)

Extra Reading: Watch

- Forward and Inverse Kinematics: Watch 3/3 of IGNOU Lectures [29min]

https://www.youtube.com/watch?v=duKD8cvtBTI

- For more clarity: Watch 12 of Addis Ababa Lectures [77 min]

[https://www.youtube.com/watch?v=NXWzk1toze4

- Robotics (13 of Addis Ababa Lectures): Inverse Kinematics [82 min]

https://www.youtube.com/watch?v=uIP3YiJLiEM

Velocity Analysis

Jacobian maps joint rates into end-effector's velocities. It depends on the manipulator configuration.

$$\text{twist of end-effector}: \mathbf{t}_e \equiv \begin{bmatrix} \boldsymbol{\omega}_e \\ \mathbf{v}_e \end{bmatrix}; \text{ Joint rates}: \dot{\boldsymbol{\theta}} = \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

$$\mathbf{t}_\mathbf{e} = \mathbf{J}\dot{\boldsymbol{\theta}} \quad \text{where } \mathbf{J} = \begin{bmatrix} \mathbf{j}_1 & \mathbf{j}_2 & \cdots & \mathbf{j}_n \end{bmatrix} \text{ and}$$

$$\mathbf{J} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_\mathbf{n} \\ \mathbf{e}_1 \times \mathbf{a}_{1e} & \mathbf{e}_2 \times \mathbf{a}_{2e} & \cdots & \mathbf{e}_\mathrm{n} \times \mathbf{a}_{ne} \end{bmatrix} \qquad ..(6.86)$$

$$\mathbf{j}_\mathrm{i} \equiv \begin{bmatrix} \mathbf{e}_\mathrm{i} \\ \mathbf{e}_\mathrm{i} \times \mathbf{a}_\mathrm{ie} \end{bmatrix}, \text{ if Joint } i \text{ is revolute} \quad \mathbf{j}_i \equiv \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_i \times \mathbf{a}_{ie} \end{bmatrix}, \text{ if Joint } i \text{ is prismatic}$$

Jacobian of a 2-link Planar Arm

$$\mathbf{J} = [\mathbf{e}_1 \times \mathbf{a}_{1e} \quad \mathbf{e}_2 \times \mathbf{a}_{2e}]$$

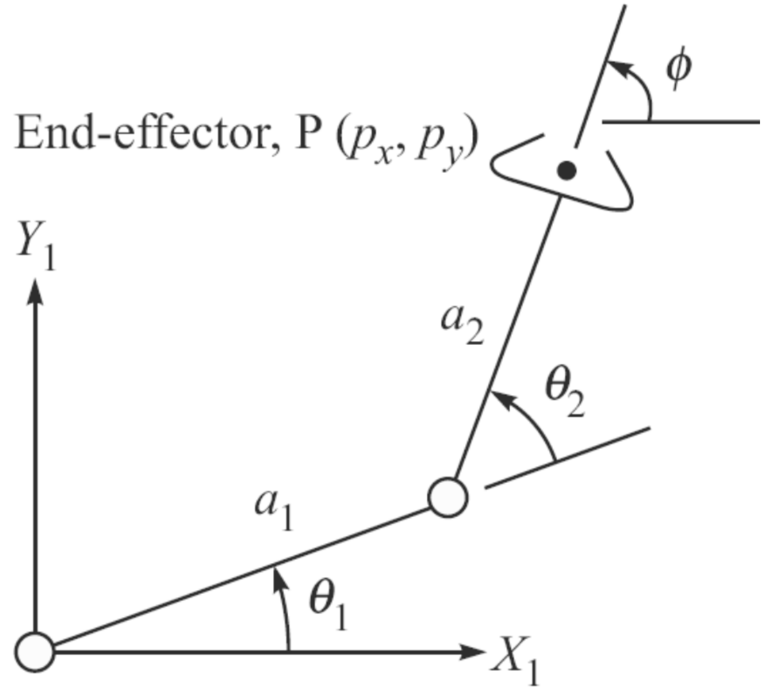
where $\mathbf{e}_1 \equiv \mathbf{e}_2 \equiv [0 \quad 0 \quad 1]^T$

$$\mathbf{a}_{1e} \equiv \mathbf{a}_1 + \mathbf{a}_2$$

$$\equiv [a_1 c_1 + a_2 c_{12} \quad a_1 s_1 + a_2 s_{12} \quad 0]^T$$

$$\mathbf{a}_{2e} \equiv \mathbf{a}_2$$

$$\equiv [a_2 c_{12} \quad a_2 s_{12} \quad 0]^T$$



Hence, $\mathbf{J} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$

Example: Singularity of 2-link RR Arm

$$\mathbf{J} \equiv \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix} \qquad \theta_2 = 0 \text{ or } \pi$$

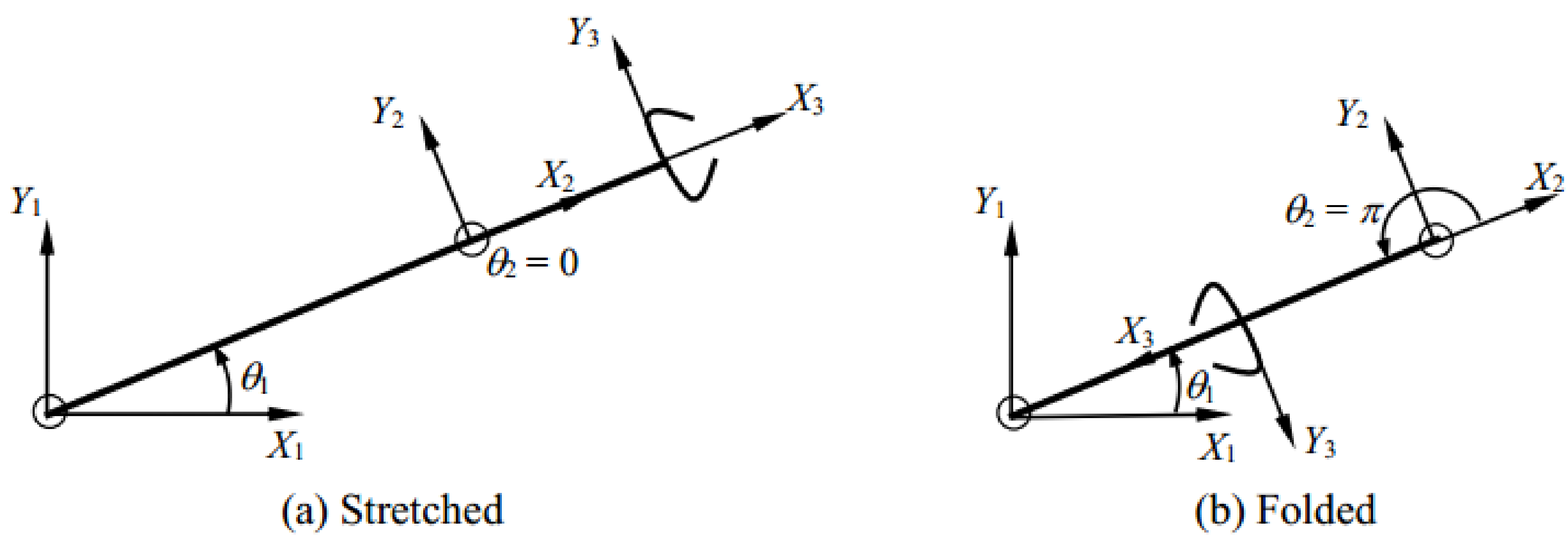


Figure 7.9 Singular configurations of a two-link planar arm

Summary

- Forward Kinematics
- Inverse kinematics
  - A spatial 6-DOF wrist-portioned has 8 solutions
- Velocity and Jacobian

THANK YOU

saha@mech.iitd.ac.in

http://sksaha.com