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Dynamics of Nonholonomic Mechanical Systems Using a Natural Orthogonal Complement

The dynamics equations governing the motion of mechanical systems composed of rigid bodies coupled by holonomic and nonholonomic constraints are derived. The underlying method is based on a natural orthogonal complement *of the matrix associated with the velocity constraint equations written in linear homogeneous form. The method is applied to the classical example of a rolling disk and an application to a 2-dof Automatic Guided Vehicle is outlined.*

1 Introduction

The theory of nonholonomic systems arose when the analytical formalism of Euler and Lagrange was found to be inapplicable to the very simple mechanical problems of rigid bodies rolling without slipping on a plane. In fact, as late as 1894, Hertz (Neĭmark and Fufaev, 1967) introduced the distinction between holonomic and nonholonomic constraints in mechanical systems. Shortly thereafter, Čaplygin (1897) derived the dynamics equations in *true coordinates*, whereas Volterra (1898) derived the equations of motion in variables which he called *motion characteristics*. Appell (1899), on the other hand, proposed a new form of the equations of motion of nonholonomic systems while introducing the concept of *acceleration energy*, *S*, similar to kinetic energy, *T*. However, in spite of the simplicity of Appell's equations, it is harder to derive expressions for *S* than it is for *T*. A few years later, Maggi (1901) showed that Volterra's and Appell's equations may be derived from his method, first proposed in 1896. More recently, Kane (Kane, 1961; Kane and Wang, 1965) introduced a method for nonholonomic systems with elimination of constraint forces.

Neĭmark and Fufaev (1967) gave the first comprehensive and systematic exposition of the mechanics of nonholonomic systems, whereas Passerello and Huston (1973) expanded Kane's formulation by eliminating the computation of acceleration components. In their method, introduction of supplementary equations similar to the constraint relations with arbitrary choice of coefficients may be difficult; furthermore, the inversion of the associated matrix is unavoidable.

With the advent of digital computation, a series of new methods in the study of mechanical systems have been developed. Huston and Passerello (1974) introduced first a computer-oriented method similar to the method of the orthogonal complement of the matrix associated with the constraint equations, which reduces the dimension of the dynamical equations by elimination of constraint forces. Later, several formulations of dynamic modeling of closed-loop mechanical systems have been reported (Paul, 1975; Wehage and Haug, 1982; Kamman and Huston, 1984; Wampler et al., 1985; Kim and Vanderploeg, 1986). Each of those formulations are applicable to holonomic and nonholonomic systems with relative advantages and disadvantages.

On the other hand, the increasing need of dynamic simulation and control of robotic mechanical systems calls for efficient computational algorithms in this respect. As a matter of fact, current research interest in robotic mechanical systems with rolling contact, such as automated guided vehicles (AGV), has renewed the interest for the modeling and simulation of nonholonomic mechanical systems (Agulló et al., 1987, 1989; Muir and Neuman, 1987, 1988).

In this paper, a new method, based on *a natural orthogonal complement* (Angeles and Lee, 1988), which has already been applied to holonomic systems (Angles and Ma, 1988; Angeles and Lee, 1989), is applied to nonholonomic systems. The idea of the orthogonal complement of velocity constraints in the derivation of dynamical equations is not new, for it has been extensively used in multibody dynamics (Huston and Passerello, 1974; Hemami and Weimer, 1981; Kamman and Huston, 1984). Orthogonal complement-based methods of dynamics analysis consist of determining a matrix—an orthogonal complement—whose columns span the nullspace of the matrix of velocity constraints. However, the said orthogonal complement is not unique. In some approaches, an orthogonal complement is found with numerical schemes which are of an intensive nature, requiring, for example, singular-value decomposition or eigenvalue computations (Wehage and Haug, 1982; Kamman and Huston, 1984). In the recent approach, the orthogonal complement comes out naturally without any complex computations. The computation of both the *natural orthogonal complement* that we use and its time derivative is outlined in Section 4. The method is illustrated with the clas-

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sical problem of the rolling disk. As well, an application to a three-wheeled 2-dof AGV is outlined.

2 Introduction of a Natural Orthogonal Complement

In this paper, as a rule, we denote vectors with boldface lower cases, while tensors and matrices with boldface upper cases, regardless of the dimensions of the vectors and matrices involved. Explicit indication of these dimensions are mentioned when defined.

As pertaining to mechanical systems composed of constrained rigid bodies, the method of analysis based on the concept of *a natural orthogonal complement*, first introduced in (Angeles and Lee, 1988), is described briefly in the following steps:

Step 1: The *twist* of the ith rigid body of the system under study, undergoing an arbitrary motion in the three-dimensional space, $\mathbf{t}_i$, is defined in terms of its angular velocity, ω_i, and the velocity of the corresponding mass center, $\dot{\mathbf{c}}_i$, both being, in general, three-dimensional vectors. Hence, $\mathbf{t}_i$ is the following six-dimensional vector:

$$\mathbf{t}_i \equiv \begin{bmatrix} \omega_i \\ \dot{\mathbf{c}}_i \end{bmatrix}. \tag{1}$$

Moreover, if $\mathbf{I}_i$ denotes the 3×3 *inertia tensor* of the ith rigid body about its mass center, and this, as well as all vector quantities involved, are referred to a coordinate system fixed to the body, then, the Newton-Euler equations governing the motion of the ith body are written as follows:

$$\mathbf{M}_i \dot{\mathbf{t}}_i = -\mathbf{W}_i \mathbf{M}_i \mathbf{t}_i + \mathbf{w}_i \tag{2}$$

where the six-dimensional *wrench* vector, $\mathbf{w}_i$, acting on the ith body is defined, in accordance with the definition of $\mathbf{t}_i$, as

$$\mathbf{w}_i \equiv \begin{bmatrix} \mathbf{n}_i \\ \mathbf{f}_i \end{bmatrix}, \tag{3}$$

$\mathbf{n}_i$ and $\mathbf{f}_i$ being three-dimensional vectors, the former denoting the resultant moment, the latter the resultant force acting at the mass center of the ith body. Now, the 3×3 Cartesian tensor $\mathbf{\Omega}_i$ is defined as

$$\mathbf{\Omega}_i \equiv \frac{\partial(\omega_i \times \mathbf{x})}{\partial \mathbf{x}} \equiv \omega_i \times \mathbf{1} \tag{4}$$

for an arbitrary three-dimensional vector $\mathbf{x}$, whereas the 6×6 matrices of *extended angular velocity*, $\mathbf{W}_i$, and of *extended mass*, $\mathbf{M}_i$, are then defined as

$$\mathbf{W}_i \equiv \begin{bmatrix} \mathbf{\Omega}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{M}_i \equiv \begin{bmatrix} \mathbf{I}_i & \mathbf{0} \\ \mathbf{0} & m_i \mathbf{1} \end{bmatrix}, \tag{5}$$

m_i, denoting the mass of the ith rigid body, whereas $\mathbf{0}$ and $\mathbf{1}$ denote the zero and the identity 3×3 tensors, respectively.

Step 2: If it is assumed that the mechanical system under study is composed of p rigid bodies, then the Newton-Euler equations for all individual bodies can be written as

$$\mathbf{M}_i \dot{\mathbf{t}}_i = -\mathbf{W}_i \mathbf{M}_i \mathbf{t}_i + \mathbf{w}_i^W + \mathbf{w}_i^N, \quad i = 1, \ldots, p \tag{6}$$

where $\mathbf{w}_i^W$ and $\mathbf{w}_i^N$ are the working wrench and the nonworking constraint wrench, both acting on the ith body, respectively. The former are understood as working moments and forces supplied by actuators or arising from dissipation; the latter, as nonworking moments and forces whose sole role is that of keeping the bodies together. Next, the $6p \times 6p$ matrices of *generalized mass*, $\mathbf{M}$, and of *generalized angular velocity*, $\mathbf{W}$, as well as the $6p$-dimensional vectors of *generalized twist*, $\mathbf{t}$, of *generalized working wrench*, $\mathbf{w}^W$, and *generalized nonworking constraint wrench*, $\mathbf{w}^N$, are defined as

$$\mathbf{M} \equiv \mathrm{diag}\,[\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_p] \tag{7}$$

$$\mathbf{W} \equiv \mathrm{diag}\,[\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_p] \tag{8}$$

$$\mathbf{t} \equiv \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_p \end{bmatrix}, \quad \mathbf{w}^W \equiv \begin{bmatrix} \mathbf{w}_1^W \\ \mathbf{w}_2^W \\ \vdots \\ \mathbf{w}_p^W \end{bmatrix}, \quad \mathbf{w}^N \equiv \begin{bmatrix} \mathbf{w}_1^N \\ \mathbf{w}_2^N \\ \vdots \\ \mathbf{w}_p^N \end{bmatrix}. \tag{9}$$

Hence, the p dynamical equations (6) can now be expressed in compact form as follows:

$$\mathbf{M}\dot{\mathbf{t}} = -\mathbf{W}\mathbf{M}\mathbf{t} + \mathbf{w}^W + \mathbf{w}^N \tag{10}$$

which is an equation formally identical to equation (2), and constitutes a set of $6p$ *unconstrained scalar* dynamical equations.

Step 3: The kinematic constraints produced by holonomic and nonholonomic couplings are derived in differential form. Within the methodology adopted here—as shown in Section 3—, every holonomic constraint gives rise to six scalar equations. As well, every nonholonomic constraint in the absence of slippage gives rise to three scalar equations. Moreover, due to the presence of the holonomic constraints, the overall constraint equations are not independent, and can be represented as a system of linear homogeneous equations on the twists. This is equivalent to the following linear homogeneous system on the vector of *generalized twist*:

$$\mathbf{A}\mathbf{t} = \mathbf{0}. \tag{11}$$

Here, $\mathbf{A}$ is a $(6\gamma + 3\nu) \times 6p$ matrix, γ and ν being the numbers of holonomic and nonholonomic couplings, respectively. Note that, with the approach introduced here, no distinction need be made between schleronomic and rheonomic constraints, for all are treated as schleronomic ones.

Step 4: Under the assumption that the degree-of-freedom of the system is n, an n-dimensional vector $\dot{\theta}$ of *independent generalized speeds* is defined. Then, the vector of generalized twist can be represented as the following linear transformation of $\dot{\theta}$:

$$\mathbf{t} = \mathbf{T}\dot{\theta} \tag{12}$$

where $\mathbf{T}$ is a $6p \times n$ matrix. Upon substitution of $\mathbf{t}$, as given by equation (12), into equation (11), and recalling that all components of $\dot{\theta}$ are independent, the following relation is readily derived:

$$\mathbf{A}\mathbf{T} = \mathbf{0} \tag{13}$$

which shows that $\mathbf{T}$ is an orthogonal complement of $\mathbf{A}$. Because of the particular form of choosing this complement—equation (12)—, $\mathbf{T}$ is termed *a natural orthogonal complement* of $\mathbf{A}$.

Step 5: By virtue of the definition of $\mathbf{A}$ and the vector of nonworking constraint wrench, the latter turns out to lie in the range of the transpose of $\mathbf{A}$ and hence, the said wrench lies in the nullspace of the transpose of $\mathbf{T}$. Therefore, upon multiplication of both sides of the $6p$-dimensional Newton-Euler uncoupled equations of the system, equation (10), by the transpose of $\mathbf{T}$, the vector of nonworking constraint wrench is eliminated from the said equation, which reduces to:

$$\mathbf{T}^T\mathbf{M}\dot{\mathbf{t}} = -\mathbf{T}^T\mathbf{W}\mathbf{M}\mathbf{t} + \mathbf{T}^T\mathbf{w}^W. \tag{14}$$

Step 6: Now, both sides of equation (12) are differentiated with respect to time, which yields

$$\dot{\mathbf{t}} = \mathbf{T}\ddot{\theta} + \dot{\mathbf{T}}\dot{\theta}. \tag{15}$$

Note that the elements of $\dot{\mathbf{T}}$ are not, in general, simply the time derivatives of the corresponding elements of $\mathbf{T}$, because the vector bases on which $\mathbf{T}$ is expressed are usually time varying.

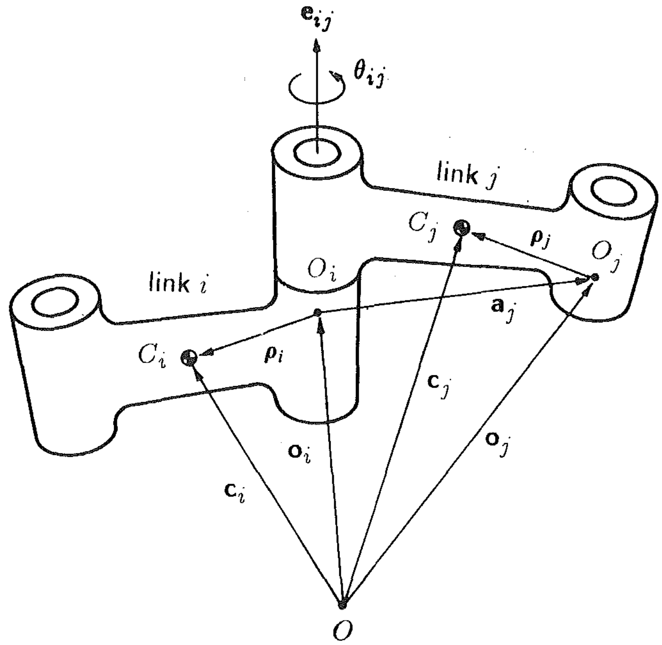

Fig. 1 Two links coupled by a revolute joint

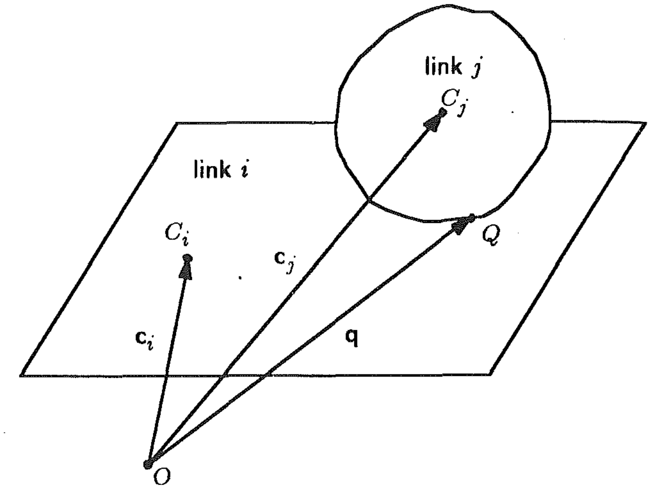

Fig. 2 A rigid body rolling on a plane

Furthermore, $\mathbf{w}^W$ is decomposed as follows:

$$\mathbf{w}^W = \mathbf{w}^A + \mathbf{w}^G + \mathbf{w}^D \tag{16}$$

where $\mathbf{w}^A$ represents the generalized wrench due to torques and forces applied by the actuators, if any, whereas $\mathbf{w}^G$ and $\mathbf{w}^D$ account for gravity and dissipative effects, respectively.

Upon substitution of equations (15) and (16) into equation (14), the following system of n-independent constrained dynamical equations are derived:

$$\mathbf{T}^T\mathbf{M}\mathbf{T}\ddot{\theta} = -\mathbf{T}^T(\mathbf{M}\dot{\mathbf{T}} + \mathbf{W}\mathbf{M}\mathbf{T})\dot{\theta} + \mathbf{T}^T(\mathbf{w}^A + \mathbf{w}^G + \mathbf{w}^D) \tag{17}$$

or,

$$\mathbf{I}(\theta)\ddot{\theta} = \mathbf{C}(\theta, \dot{\theta})\dot{\theta} + \tau + \gamma + \delta \tag{18}$$

where

$\mathbf{I} \equiv \mathbf{T}^T\mathbf{M}\mathbf{T}$ $\equiv$ $n \times n$ matrix of generalized inertia.
$\mathbf{C} \equiv -\mathbf{T}^T(\mathbf{M}\dot{\mathbf{T}} + \mathbf{W}\mathbf{M}\mathbf{T})$ $\equiv$ $n \times n$ matrix of convective inertia terms.
$\tau \equiv \mathbf{T}^T\mathbf{w}^A$ $\equiv$ n-dimensional vector of generalized driving force.
$\gamma \equiv \mathbf{T}^T\mathbf{w}^G$ $\equiv$ n-dimensional vector of generalized force due to gravity.
$\delta \equiv \mathbf{T}^T\mathbf{w}^D$ $\equiv$ n-dimensional vector of generalized dissipative force.

From the foregoing discussion, then, it becomes apparent that equation (18) represents the system's Euler-Lagrange dynamical equations, free of nonworking generalized constraint forces.

3 Derivation of the Kinematic Constraints

Mechanical couplings that produce holonomic and nonholonomic constraints on the twists—velocity constraints—of the coupled bodies, say $\mathbf{t}_i$ and $\mathbf{t}_j$ defined as in equation (1), have the following form:

$$\mathbf{A}_i\mathbf{t}_i + \mathbf{A}_j\mathbf{t}_j = 0. \tag{19}$$

The foregoing constraint equations are seen to be *linear homogeneous* in the twists of the coupled bodies. Moreover, the coefficient matrices $\mathbf{A}_i$ and $\mathbf{A}_j$ are, in general, configuration dependent.

3.1 Holonomic Systems. In the case of a holonomic coupling, for example, a revolute pair, the constraint equations are readily derived as follows: If $\dot{\theta}_{ij}$ is the joint rate for the revolute coupling between the ith and the jth links, then, referring to Fig. 1, the relative angular velocity of the jth link with respect to the ith link, $\omega_j - \omega_i$, is $\dot{\theta}_{ij}\mathbf{e}_{ij}$. Thus, the equation constraining the angular velocities of two successive links is the following:

$$\mathbf{e}_{ij} \times (\omega_j - \omega_i) = 0. \tag{20}$$

Furthermore, from Fig. 1 it is clear that

$$\dot{\mathbf{c}}_j = \dot{\mathbf{c}}_i + \omega_j \times (\mathbf{a}_j + \rho_j) - \omega_i \times \rho_i. \tag{21}$$

Equations (20) and (21) are now written in terms of the link twists, which readily produces an equation of the form of equation (19), where the 6×6 matrices $\mathbf{A}_i$ and $\mathbf{A}_j$ are as shown below:

$$\mathbf{A}_i = \begin{bmatrix} -\mathbf{e}_{ij} \times \mathbf{1} & \mathbf{0} \\ -\rho_i \times \mathbf{1} & -\mathbf{1} \end{bmatrix}, \quad \mathbf{A}_j = \begin{bmatrix} \mathbf{e}_{ij} \times \mathbf{1} & \mathbf{0} \\ (\mathbf{a}_j + \rho_j) \times \mathbf{1} & \mathbf{1} \end{bmatrix}. \tag{22}$$

Mechanical couplings that produce holonomic constraints other than the revolute pair are, e.g., the prismatic pair and gear trains. The pulley-belt and the cam-follower transmissions produce holonomic constraints as well. The different forms of the 6×6 matrices $\mathbf{A}_i$ and $\mathbf{A}_j$ for different couplings can be obtained in a similar way, as described in Angeles and Lee (1989).

3.2 Nonholonomic Systems. We will limit the discussion to nonholonomic pure rolling, and hence, no slippage is considered. As an example of this type of nonholonomic coupling, a rigid body rolling without slipping on a plane is shown in Fig. 2, the nonslip condition being derived below. If the rigid body is considered as the jth body of a system and the plane is a part of the boundary of the ith body, then, referring to Fig. 2, the arising nonholonomic constraint is stated as

$$\dot{\mathbf{c}}_j = \dot{\mathbf{c}}_i + \omega_j \times (\mathbf{c}_j - \mathbf{q}) - \omega_i \times (\mathbf{c}_i - \mathbf{q}) \tag{23}$$

where $\mathbf{q}$ is the position vector of the contact point Q. Moreover, equation (23) can also be written as equation (19), with the 3×6 matrices $\mathbf{A}_i$ and $\mathbf{A}_j$ defined as:

$$\mathbf{A}_i = [-\mathbf{C}_i \ \ -\mathbf{1}], \quad \mathbf{A}_j = [\mathbf{C}_j \ \ \mathbf{1}] \tag{24}$$

and the Cartesian tensors $\mathbf{C}_i$ and $\mathbf{C}_j$ defined as follows:

$$\mathbf{C}_i = (\mathbf{c}_i - \mathbf{q}) \times \mathbf{1}, \quad \text{and} \quad \mathbf{C}_j = (\mathbf{c}_j - \mathbf{q}) \times \mathbf{1}. \tag{25}$$

From equations (22) and (24) it is apparent that a kinematic constraint, whether holonomic or nonholonomic, can be written in the form of a linear homogeneous equation involving the twists of every pair of coupled bodies of a mechanical system. An essential difference between the holonomic and the nonholonomic constraints previously derived is pointed out here. Whereas the former lead to six-dimensional constraint equations, the latter lead to three-dimensional equations. As one can readily verify, the holonomic constraints derived above are not independent, although the nonholonomic constraints are. If slippage is possible, then the number of independent nonholonomic constraint equations diminishes, for the relative motion gains degrees-of-freedom. Furthermore, if the kinematic constraints introduced by the couplings comprise γ holonomic and ν nonholonomic couplings, then the arising kinematic constraint equations can be described in compact form as in equation (11), where $\mathbf{A}$ is a $(6\gamma + 3\nu) \times 6p$ matrix. This approach departs from that reported in the literature of deriving *independent* scalar constraint equations that are linear in the body twists, but *not necessarily homogeneous*. In fact, the homogeneity of these constraint equations allows us to regard all constraints as schleronomic.

4 Calculation of T and $\dot{\mathbf{T}}$

The derivation of the *natural orthogonal complement* from equation (12) is, in general, costly, except for simple systems, as in the example given below. An efficient method of calculating $\mathbf{T}$ numerically (Ma and Angeles, 1989) can be readily derived by noticing that $\mathbf{T}$ depends on *generalized coordinates* only. Moreover, the jth column of $\mathbf{T}$ equals $\partial \mathbf{t}/\partial \dot{\theta}_j$, for $j = 1, 2, \ldots, n$. Thus, $\mathbf{T}$ can be found as follows:

$$\mathbf{T} = [\mathbf{t}|_{\dot{\theta}_1=1}, \mathbf{t}|_{\dot{\theta}_2=1}, \ldots, \mathbf{t}|_{\dot{\theta}_n=1}]_{\text{other } \dot{\theta}\text{'s of } \dot{\theta} \text{ are zero}}, \quad (26)$$

i.e., the jth column of $\mathbf{T}$ is calculated as the generalized twist of the system assuming that all the independent speeds are zero but the jth one, which has a value of unity.

In simulation applications, there is no need of an explicit evaluation of $\dot{\mathbf{T}}$. In fact, the generalized inertia terms that are quadratic in $\dot{\theta}$, $\mathbf{C}(\theta, \dot{\theta})\dot{\theta}$, of equation (18), can be evaluated efficiently with the technique introduced by Walker and Orin (1982) for serial manipulators and which has been extended by Ma and Angeles (1989) for parallel manipulators. This is given as follows:

$$\mathbf{C}(\theta, \dot{\theta})\dot{\theta} = \tau + \gamma + \delta|_{\ddot{\theta}=0}, \quad (27)$$

i.e., the entire first term of equation (18) is computed from *inverse dynamics* (Walker and Orin, 1982) as the force, consisting of driving, gravity, and dissipative force, if any, respectively, required to maintain the system's current joint displacements and velocities, but with zero accelerations $\ddot{\theta}$.

5 Example: Disk Rolling on an Inertial Plane

Shown in Fig. 3 is a disk rolling on a plane that is considered fixed to an inertial frame. Then, the mass center of the disk, C, is assumed to be coincident with its centroid, the velocity of the latter being denoted by $\dot{\mathbf{c}}$. Moreover, vector $\dot{\mathbf{c}}$ is related to the disk's angular velocity, ω, as follows:

$$\dot{\mathbf{c}} = \omega \times (\mathbf{c} - \mathbf{q}) \quad (28)$$

where $\mathbf{c}$ and $\mathbf{q}$ are position vectors of the mass center C and the point of contact Q, respectively. Since p, the number of moving rigid bodies, is equal to one, i.e., the disk, the six-dimensional vector of twist, $\mathbf{t}$, similar to equation (12), can be written as a linear transformation of the *independent generalized speeds*. We can choose, for example, the angular velocity ω of the disk as the vector of independent generalized speeds, because the degree-of-freedom of the system is three. Now, $\mathbf{t}$ is given by

$$\mathbf{t} = \mathbf{T}\omega \quad (29)$$

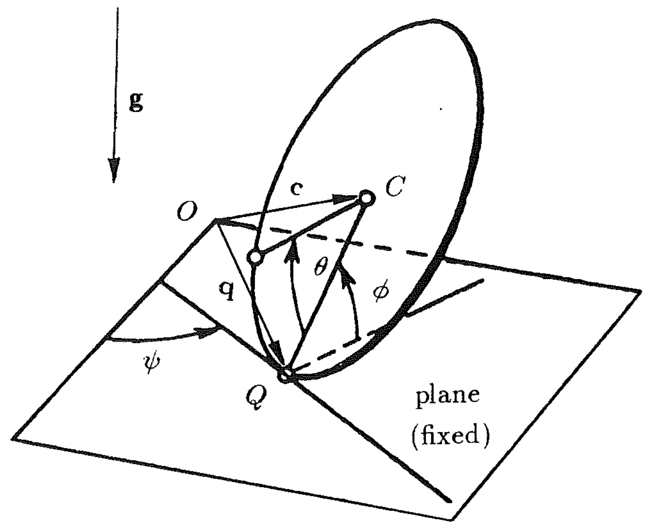

Fig. 3 A disk rolling on an inertial plane

with the 6×3 matrix $\mathbf{T}$ defined as:

$$\mathbf{T} = \begin{bmatrix} \mathbf{1} \\ (\mathbf{q}-\mathbf{c})\times\mathbf{1} \end{bmatrix}. \quad (30)$$

Next, the constraint equation is simply derived as:

$$\mathbf{At} = \mathbf{0} \quad (31)$$

where

$$\mathbf{A} = [(\mathbf{c}-\mathbf{q})\times\mathbf{1} \;\; \mathbf{1}]$$

and hence, $\mathbf{A}$ is a 3×6 matrix. Matrix $\mathbf{T}$ is an orthogonal complement of $\mathbf{A}$, which can easily be proven by simply performing the product $\mathbf{AT}$.

Now, the 6×6 *generalized mass and generalized angular velocity* matrices $\mathbf{M}$ and $\mathbf{W}$ are defined as follows:

$$\mathbf{M} = \begin{bmatrix} \mathbf{I}_C & \mathbf{0} \\ \mathbf{0} & m\mathbf{1} \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} \omega\times\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where m and $\mathbf{I}_C$ are the mass and the 3×3 inertia tensor about the mass center of the disk, respectively. To obtain the equations of motion, $\dot{\mathbf{T}}$ is calculated as:

$$\dot{\mathbf{T}} = \begin{bmatrix} \mathbf{0} \\ -\dot{\mathbf{c}}\times\mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -[\omega\times(\mathbf{c}-\mathbf{q})]\times\mathbf{1} \end{bmatrix}.$$

Then, the 3×3 *generalized inertia* matrix, $\mathbf{I}$, as given in equation (18), is derived as:

$$\mathbf{I} = [\mathbf{1} \; -(\mathbf{q}-\mathbf{c})\times\mathbf{1}] \begin{bmatrix} \mathbf{I}_C & \mathbf{0} \\ \mathbf{0} & m\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ (\mathbf{q}-\mathbf{c})\times\mathbf{1} \end{bmatrix}$$

$$= \mathbf{I}_C + m[(\mathbf{q}-\mathbf{c})\cdot(\mathbf{q}-\mathbf{c})\mathbf{1} - (\mathbf{q}-\mathbf{c})\otimes(\mathbf{q}-\mathbf{c})] \quad (32)$$

where $\otimes$ denotes the *tensor product* of the two vectors beside it. Equation (32) has the following interpretation: The calculated matrix of *generalized inertia* $\mathbf{I}$ is nothing but the moment of inertia of the disk about the contact point Q. Moreover, the matrix $\mathbf{C}$ of convective inertia terms defined in equation (18), is derived as follows:

$$\mathbf{WMT} = \begin{bmatrix} \omega\times\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I}_C\mathbf{0} & \mathbf{0} \\ & m\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ (\mathbf{q}-\mathbf{c})\times\mathbf{1} \end{bmatrix} = \begin{bmatrix} \omega\times\mathbf{I}_C \\ \mathbf{0} \end{bmatrix}$$

and $\mathbf{M}\dot{\mathbf{T}}$ =

$$\begin{bmatrix} \mathbf{I}_C & \mathbf{0} \\ \mathbf{0} & m\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ -[\omega\times(\mathbf{q}-\mathbf{c})]\times\mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -m[\omega\times(\mathbf{q}-\mathbf{c})]\times\mathbf{1} \end{bmatrix}.$$

Therefore,

$$\mathbf{C} = -\mathbf{T}^T(\mathbf{WMT} + \mathbf{M}\dot{\mathbf{T}})$$
$$= -\omega\times\mathbf{I}_C - m[(\mathbf{q}-\mathbf{c})\times\omega]\otimes(\mathbf{q}-\mathbf{c}). \quad (33)$$

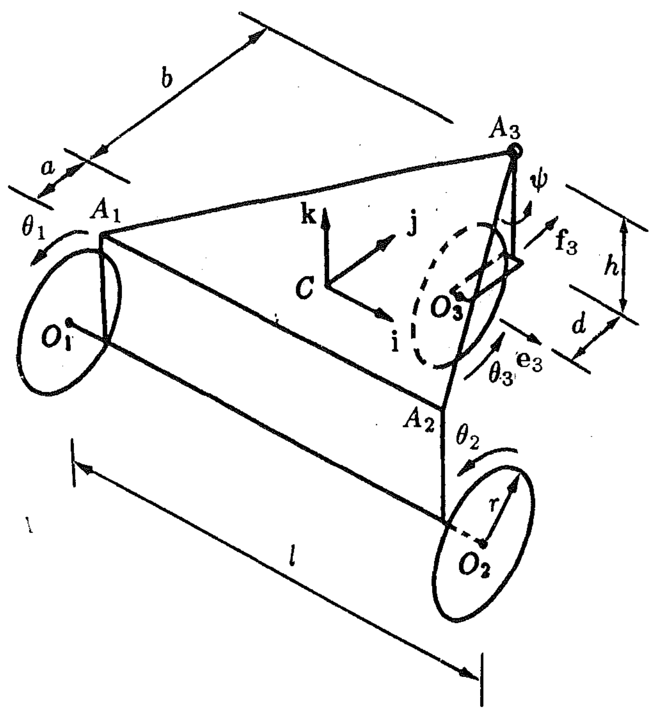

Fig. 4 Three-dimensional view of a 2-dof AGV

If dissipation is neglected and there is no actuation, the three-dimensional vectors $\boldsymbol{\delta}$ and $\boldsymbol{\tau}$ vanish but $\boldsymbol{\gamma}$, the *generalized force* due to gravity, is given as:

$$\boldsymbol{\gamma} = \mathbf{T}^T \mathbf{w}^W = [\mathbf{1} - (\mathbf{q} - \mathbf{c}) \times \mathbf{1}] \begin{bmatrix} \mathbf{0} \\ m\mathbf{g} \end{bmatrix} = m(\mathbf{c} - \mathbf{q}) \times \mathbf{g} \qquad (34)$$

where $\mathbf{g}$ is the vector of gravity acceleration. Thus, $\boldsymbol{\gamma}$ is nothing but the moment of the gravity force about the contact point Q.

Finally, the equation of motion of the disk rolling on a plane without slipping can be written as:

$$\mathbf{I}\boldsymbol{\omega} = \mathbf{C}\boldsymbol{\omega} + \boldsymbol{\gamma} \qquad (35)$$

where $\mathbf{I}$, $\mathbf{C}$, and $\boldsymbol{\gamma}$ are as derived in equations (32), (33) and (34), respectively.

6 Application to a 2-DOF Automatic Guided Vehicle: An Outline

The kinematics and dynamics analyses of a three-wheeled 2-dof AGV, using the natural orthogonal complement introduced above, have been discussed in detail in (Saha and Angeles, 1989). An outline of the modeling of the same vehicle is included here for illustration purposes.

Step 1: It is assumed that the AGV under study, as shown in Fig. 4, contains a platform, two rear driving wheels, and a caster wheel in the front. Here, note that not all joint motions are actuated or driven by motors. Only the two rear wheels are actuated by independent motors whose motions are grouped in the two-dimensional vector of actuated joint motions, $\boldsymbol{\theta}_a = [\theta_1, \theta_2]^T$, in agreement with the fact that the vehicle has two degrees-of-freedom. The other two joint motions, spinning of the front wheel, θ_3, and rotation of the bracket holding the front caster wheel with respect to the platform, ψ, are unactuated joint motions. Thus, the angular displacements of the actuated joints can be considered as the independent joint variables and their time rates as the independent speeds. Now, considering the motion of the vehicle being planar, the three-dimensional *reduced twist* vector, $\mathbf{t}'_p = [\omega, v_i, v_j]^T$, is defined, where ω denotes the scalar angular velocity of the platform, while v_i and v_j are the velocity components of mass-center of the platform in the directions of $\mathbf{i}$ and $\mathbf{j}$, respectively. Vector $\mathbf{t}'_p$ is now written as a linear transformation of the two-dimensional vector of actuated joint rates, $\dot{\boldsymbol{\theta}}_a$. This relation, which was derived in (Saha and Angeles, 1989), is given as

$$\mathbf{t}'_p = \mathbf{T}'_p \dot{\boldsymbol{\theta}}_a \qquad (36)$$

where $\mathbf{T}'_p$ is a 3×2 matrix.

Step 2: For inverse kinematics, $\dot{\boldsymbol{\theta}}_a$ is to be calculated for given $\mathbf{t}_p'$. Now, for kinematically admissible motions, it can be proven that $\mathbf{t}_p'$ lies in the range of $\mathbf{T}_p'$. Therefore, the three components of $\mathbf{t}'_p$ cannot be supplied arbitrarily, which is evidenced by the fact that the AGV has only two degrees-of-freedom. Hence, the *actuated joint-rate* vector, $\dot{\boldsymbol{\theta}}_a$, can be calculated as the least-square *approximation* of the overdetermined system of equations, appearing in equation (36), with zero error. This means that the least-square approximation of that system is, in fact, its solution, for this system is overdetermined only formally.

Step 3: The relation in equation (36) is differentiated with respect time. This is clearly as follows:

$$\dot{\mathbf{t}}_p' = \mathbf{T}'_p \ddot{\boldsymbol{\theta}}_a + \dot{\mathbf{T}}_p' \dot{\boldsymbol{\theta}}_a. \qquad (37)$$

Then, once the solution for $\dot{\boldsymbol{\theta}}_a$ is known from Step 2, $\ddot{\boldsymbol{\theta}}_a$ can be solved from the equation (37) following exactly the same least-square approach of Step 2.

Step 4: To obtain the two-dimensional *actuated joint-angle* vector, $\boldsymbol{\theta}_a$, the expression for $\dot{\boldsymbol{\theta}}_a$ obtained from equation (36) is integrated, with known initial conditions, by any standard integration scheme.

Step 5: The dynamical equations of motion, which are derived in Saha and Angeles (1989), can now be written as:

$$\mathbf{I}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}_a = \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a)\dot{\boldsymbol{\theta}}_a + \boldsymbol{\tau}_a \qquad (38)$$

where $\mathbf{I}$ and $\mathbf{C}$ are 2×2 matrices of *generalized inertia* and of *convective inertia terms*, respectively. Moreover, $\boldsymbol{\tau}_a$ is a two-dimensional vector of *generalized torque* supplied by the actuators, $\boldsymbol{\theta}$ is the four-dimensional vector of actuated and unactuated joint angles, whereas $\dot{\boldsymbol{\theta}}_a$ and $\ddot{\boldsymbol{\theta}}_a$ are the two-dimensional vectors of actuated joint rates and accelerations, respectively.

Step 6: With the above mentioned methodology, the simulation of the automatic guided vehicle under study, when traversing a circular path (Saha and Angeles, 1989) and two parallel straight lines connected with a smooth curve, has been performed, the results not being included here due to space limitations.

7 Conclusions

In this paper, the derived dynamical model of mechanical systems with nonholonomic and holonomic constraints, equation (18), using *a natural orthogonal complement*, were derived free of constraint forces, which usually appear when the Newton-Euler method is applied to derive the governing equations. Moreover, what the derivation of the dynamical equations presented here shows is that these can be obtained without resorting to lengthy partial differentiations, which would be the case if either a straightforward or a recursive derivation of the Euler-Lagrange equations were attempted. From an algorithmic point of view, the method of the *natural orthogonal complement* has advantages over other, similar methods, as pointed out in Section 4. Here, $\mathbf{T}$ is calculated efficiently by avoiding eigenvalue or singular-value calculations. Moreover, in simulation applications the convective inertia terms, which involve the evaluation of $\dot{\mathbf{T}}$, as well as a few more matrix-times-vector multiplications and vector additions, is calculated very efficiently using Walker and Orin's scheme (Walker and Orin, 1982). On the other hand, the direct application of Kane's method (Kane, 1961; Kane and Wang, 1965) requires the evaluation of acceleration terms, where the method of Passerello and Huston (1973) requires a matrix inversion. None of these is needed when *a natural orthogonal complement* is used.

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