The $\mathbf{UDU}^T$ Decomposition of Manipulator Inertia Matrix

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Abstract

In this paper, the $\mathbf{UDU}^T$ *decomposition of the generalized inertia matrix of an n-link serial manipulator is presented in symbolic form, where* $\mathbf{U}$ *and* $\mathbf{D}$*, respectively, are the upper triangular and diagonal matrices. To render the decomposition, the elementary upper triangular matrices, associated to a modified Gaussian elimination, are introduced, whereas each element of the inertia matrix is written as an expression, instead of finding it as a number with the aid of an algorithm. The resulting* $\mathbf{UDU}^T$ *decomposition shows recursive relations among the elements of the associated matrices. Thus, algorithms of order 'n' can be developed not only for the inverse but also for the forward dynamics. As an illustration, a forward dynamics algorithm is presented here.*

1 Introduction

In robotics, the inverse and forward dynamics of robotic manipulators, associated to their control and simulation, respectively, are well-known problems. While, based on Euler-Lagrange equations [1], Newton-Euler equations [2], and other techniques, e.g., [3], many efficient recursive inverse dynamics algorithms of order n, $\mathcal{O}(n)$, i.e., whose complexities are linear in the number of links, n, exist, an $\mathcal{O}(n)$ algorithm for forward dynamics was not available until very recently [4, 5]. The reason lies in the difficulties of deriving the $n \times n$ generalized inertia matrix (GIM) of the manipulator, and the solution of the set of n linear algebraic equations in joint accelerations, namely, the dynamic equations of motion. If a straightforward approach is undertaken, i.e., an algorithm is used to calculate the GIM and decompose it numerically, using, for example, the Cholesky decomposition [6], before the joint accelerations are solved by forward and backward substitutions, the complexity of $\mathcal{O}(n^3)$ is inevitable.

Hence, a different look into the problem was sought, which resulted in an approach called *articulated-body inertia* [4]. The new concept allowed to calculate the joint accelerations with $\mathcal{O}(n)$ computations. Later, in [5] and consequent publications, $\mathcal{O}(n)$ forward dynamics algorithms for different robotic systems are reported, which are based on Kalman filtering and smoothing techniques, arising in the state estimation theory. This approach provides a deeper insight to the manipulator dynamics. It was possible due to the establishment of the equivalency of the discrete-time state space systems to the spatially recursive state space model in which the distance between two successive joints plays the role of time interval of the discrete-time models.

In this paper, the elements of the GIM are derived as *expressions*, as opposed to *numbers* in an algorithmic approach. Such representation enables us to perform the Gaussian elimination (GE) [6] of the GIM symbolically. Note, however, the modifications in the introduced GE, as carried out in §**A**, which is called here the *reverse* Gaussian elimination (RGE). The symbolic RGE results in the desired $\mathbf{UDU}^T$ decomposition of the GIM, where $\mathbf{U}$ and $\mathbf{D}$ are the upper triangular and the diagonal matrices, respectively, whose elements have symbolic recursive representations. The recursiveness is not recognizable in the algorithmic approach, where the operations are performed on numbers. Thus, using the proposed approach, $\mathcal{O}(n)$ algorithms for both the inverse and forward dynamics of the system at hand can be developed.

As an illustration of the proposed decomposition, an $\mathcal{O}(n)$ forward dynamics algorithm is presented in §**B**. The efficiency of the scheme, compared to that of an efficient (n^3) algorithm, e.g., [7], is appreciable while $n \geq 12$, as in [4]. Thus, the proposed forward dynamics scheme is suitable only for highly redundant manipulators ($n \geq 12$), which may be required to

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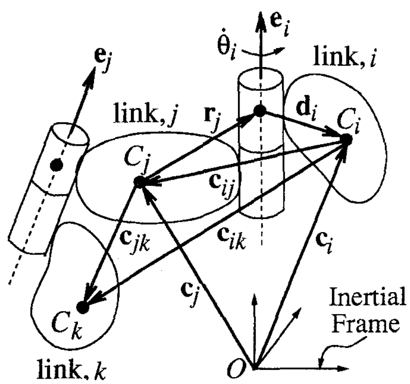

Fig. 1 A system of coupled links.

achieve non-stop operation in the presence of actuator failures, to maneuver in a highly constrained environment, etc. The decomposition, due to the availability of the recursive symbolic relations, is, nevertheless, very useful for analytic investigation of the robot dynamics or to verify the intermediate steps of the forward dynamics calculations.

2 Decoupled NOC

Since the natural orthogonal complement (NOC)—a matrix that relates the velocities of the links of a mechanical system to its joint rates, and used to reduce the Newton-Euler equations of motion of uncoupled links of the system to an independent set of Euler-Lagrange equations—in its present form, i.e., as obtained in [8, 9], is not suitable to write the elements of the GIM as expressions, it is derived here as two *decoupled* matrices. This allows to perform the symbolic RGE, as done in §4. Now, let us define the 6-dimensional vector of the *twist*, or the *spatial velocity* [4, 5], of the ith link, Fig. 1, as

$$\mathbf{t}_i \equiv [\boldsymbol{\omega}_i^T,\ \dot{\mathbf{c}}_i^T]^T \tag{1}$$

where $\boldsymbol{\omega}_i$ and $\dot{\mathbf{c}}_i$ are the 3-dimensional vectors of angular velocity and the velocity of the mass center of the ith link, C_i, respectively. Moreover, for the system of coupled links, Fig. 1, $\boldsymbol{\omega}_i$ and $\dot{\mathbf{c}}_i$ are written as

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_j + \dot\theta_i \mathbf{e}_i \tag{2a}$$

$$\dot{\mathbf{c}}_i = \dot{\mathbf{c}}_j + \boldsymbol{\omega}_j \times \mathbf{r}_j + \boldsymbol{\omega}_i \times \mathbf{d}_i \tag{2b}$$

where $\boldsymbol{\omega}_j$ and $\dot{\mathbf{c}}_j$ are the angular velocity and the velocity of C_j of the jth link, respectively, whereas the Cartesian vectors, $\mathbf{r}_j$ and $\mathbf{d}_i$, are shown in Fig. 1. Combining eqs.(2a) and (2b), the twist, $\mathbf{t}_i$, is expressed as a function of $\mathbf{t}_j$ and $\dot\theta_i$, i.e.,

$$\mathbf{t}_i = \mathbf{B}_{ij}\mathbf{t}_j + \mathbf{p}_i\dot\theta_i \tag{3}$$

where the 6×6 matrix, $\mathbf{B}_{ij}$, and the 6-dimensional vector, $\mathbf{p}_i$, are given by

$$\mathbf{B}_{ij} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{C}_{ij} & \mathbf{1} \end{bmatrix} \quad \text{and} \quad \mathbf{p}_i \equiv \begin{bmatrix} \mathbf{e}_i \\ \mathbf{e}_i \times \mathbf{d}_i \end{bmatrix} \tag{4}$$

$\mathbf{1}$ and $\mathbf{O}$ being the 3×3 identity and zero matrices, respectively, which, henceforth, should be understood as of dimensions compatible to the size of a matrix where they appear. Moreover, $\mathbf{C}_{ij}$ is the 3×3 cross-product tensor, associated to the vector, $\mathbf{c}_{ij} = \mathbf{c}_j - \mathbf{c}_i$. A 3×3 cross-product tensor, associated to the 3-dimensional vector, $\mathbf{z}$, denoted by $\mathbf{Z}$, is defined by

$$\mathbf{Z} \equiv \mathbf{z} \times \mathbf{1} \equiv \frac{\partial(\mathbf{z} \times \mathbf{x})}{\partial \mathbf{x}} \tag{5}$$

for any arbitrary 3-dimensional vector, $\mathbf{x}$. Furthermore, $\mathbf{e}_i$ is the unit vector parallel to the axis of the ith revolute pair.

It is pointed out here that the matrix, $\mathbf{B}_{ij}$, and the vector, $\mathbf{p}_i$, have the following interpretations: while the former multiplied to $\mathbf{t}_j$ gives $\mathbf{t}_i$, if there is no ith joint, the latter takes into account the effect of the ith joint motion. Also, from the definition of $\mathbf{B}_{ij}$, eq.(4), and Fig. 1, the following properties are derived:

$$\mathbf{B}_{ij}\mathbf{B}_{jk} = \mathbf{B}_{ik} \quad \text{and} \quad \mathbf{B}_{ii} = \mathbf{1} \tag{6}$$

Now, for the manipulator consisting of n links, as denoted in Fig. 2 by #1, $\cdots$, #n, coupled by n revolute pairs, namely, $1, \cdots, n$, the $6n$-dimensional generalized twist, $\mathbf{t}$, is defined as

$$\mathbf{t} \equiv [\mathbf{t}_1^T,\ \cdots,\ \mathbf{t}_n^T]^T \tag{7}$$

where $\mathbf{t}_i$, for $i = 1, \cdots, n$, is given in eq.(1). Using eq.(3) and the properties given by eq.(6), vector $\mathbf{t}$ is represented as

$$\mathbf{t} = \mathbf{T}\dot{\boldsymbol{\theta}}, \qquad \text{where} \quad \mathbf{T} \equiv \mathbf{T}_l\mathbf{T}_d \tag{8}$$

In eq.(8), the $6n \times n$ matrix, $\mathbf{T}$, is the natural orthogonal complement (NOC) of the system at hand, where $\mathbf{T}_l$ and $\mathbf{T}_d$ are the $6n \times 6n$ lower block triangular and the $6n \times n$ block diagonal matrices, respectively, i.e.,

$$\mathbf{T}_l \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{B}_{21} & \mathbf{1} & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{n1} & \mathbf{B}_{n2} & \cdots & \mathbf{1} \end{bmatrix} \tag{9a}$$

$$\mathbf{T}_d \equiv \text{diag}[\mathbf{p}_1,\ \cdots,\ \mathbf{p}_n] \tag{9b}$$

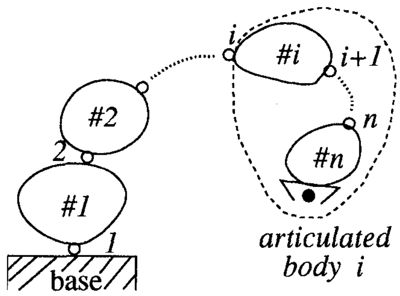

Fig. 2 An n-link n-DOF manipulator.

whereas the n-dimensional vector of joint rates, $\dot{\boldsymbol{\theta}}$, is as follows:

$$\dot{\boldsymbol{\theta}} \equiv [\dot{\theta}_1, \cdots, \dot{\theta}_n]^T \tag{9c}$$

Note, the foregoing derivations are also possible for manipulators with joints other than revolute, e.g., prismatic, in which the matrix, $\mathbf{T}_d$, will change.

3 Generalized Inertia Matrix

If $\dot{\boldsymbol{\theta}}$ of eq.(8) denotes the vector of independent generalized speeds of the manipulator under study, Fig. 2, the $n \times n$ generalized inertia matrix (GIM), $\mathbf{I}$, as defined in [8, 9], is given as

$$\mathbf{I} \equiv \mathbf{T}^T\mathbf{M}\mathbf{T} \equiv \mathbf{T}_d^T\mathbf{T}_l^T\mathbf{M}\mathbf{T}_l\mathbf{T}_d \tag{10}$$

where the $6n \times 6n$ generalized mass matrix, $\mathbf{M}$, is defined by

$$\mathbf{M} \equiv \text{diag}[\mathbf{M}_1,\ \cdots,\ \mathbf{M}_n] \tag{11}$$

The 6×6 matrix, $\mathbf{M}_i$, for $i = 1, \cdots, n$, being the *extended mass* of the ith link with respect to its mass center, C_i, which is also referred to as the *spatial inertia matrix* [4, 5]. Matrix $\mathbf{M}_i$ is

$$\mathbf{M}_i \equiv \begin{bmatrix} \mathbf{I}_i & \mathbf{O} \\ \mathbf{O} & m_i\mathbf{1} \end{bmatrix} \tag{12}$$

where m_i and $\mathbf{I}_i$ are the mass and the 3×3 inertia tensor about, C_i, of the ith link, respectively. Upon substitution of eq.(9a) into eq.(10), the GIM, $\mathbf{I}$, is rewritten as

$$\mathbf{I} \equiv \mathbf{T}_d^T\tilde{\mathbf{M}}\mathbf{T}_d, \qquad \text{where} \quad \tilde{\mathbf{M}} \equiv \mathbf{T}_l^T\mathbf{M}\mathbf{T}_l \tag{13}$$

Matrix $\tilde{\mathbf{M}}$ is the $6n \times 6n$ symmetric matrix, i.e.,

$$\tilde{\mathbf{M}} \equiv \begin{bmatrix} \tilde{\mathbf{M}}_1 & \mathbf{B}_{21}^T\tilde{\mathbf{M}}_2 & \cdots & \mathbf{B}_{n1}^T\tilde{\mathbf{M}}_n \\ \tilde{\mathbf{M}}_2\mathbf{B}_{21} & \tilde{\mathbf{M}}_2 & \cdots & \mathbf{B}_{n2}^T\tilde{\mathbf{M}}_n \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathbf{M}}_n\mathbf{B}_{n1} & \tilde{\mathbf{M}}_n\mathbf{B}_{n2} & \cdots & \tilde{\mathbf{M}}_n \end{bmatrix} \tag{14}$$

where the 6×6 matrix, $\tilde{\mathbf{M}}_i$, for $i = 1, \cdots, n$, is as follows:

$$\tilde{\mathbf{M}}_i \equiv \mathbf{M}_i + \tilde{\mathbf{M}}_{i,i+1} \tag{15a}$$

in which the matrix, $\tilde{\mathbf{M}}_{i,i+1}$, is defined by

$$\tilde{\mathbf{M}}_{i,i+1} \equiv \sum_{l=i+1}^{n} \mathbf{B}_{li}^T\mathbf{M}_l\mathbf{B}_{li} \tag{15b}$$

that can be calculated recursively as

$$\tilde{\mathbf{M}}_{i,i+1} = \mathbf{B}_{i+1,i}^T\tilde{\mathbf{M}}_{i+1}\mathbf{B}_{i+1,i} \tag{15c}$$

In eq.(15c), if $i = n$, $\tilde{\mathbf{M}}_{n+1} = \mathbf{O}$, because there is no $(n+1)$st link in the kinematic chain. Hence, $\tilde{\mathbf{M}}_{n,n+1}$ vanish, and from eq.(15a), $\tilde{\mathbf{M}}_n \equiv \mathbf{M}_n$.

The interpretations, associated to matrix $\tilde{\mathbf{M}}_i$, eq.(15a), are provided below:

(A) For $i = n$, $\tilde{\mathbf{M}}_n \equiv \mathbf{M}_n$, and the expression, $\mathbf{M}_n\mathbf{t}_n$, denotes the 6-dimensional vector, whose first three components are the angular momentum of the nth link about C_n, and the rest are its linear momentum, which is obvious from eqs.(1) and (12).

(B) For $i = n - 1$, using eqs.(15a) and (15c), the expression, $\tilde{\mathbf{M}}_{n-1}\mathbf{t}_{n-1}$, can be verified as the 6-dimensional momenta vector of the system consisting of two rigidly connected links, $\#(n-1)$ and $\#n$, i.e., *composite body* $(n-1)$, whose angular momentum is calculated about C_{n-1}. Thus, similar to the definition of $\mathbf{M}_n$, matrix $\tilde{\mathbf{M}}_{n-1}$ is the extended mass of *composite body* $(n-1)$ with respect to C_{n-1}.

(C) Extending the explanations, (B), to $\tilde{\mathbf{M}}_i$, matrix $\tilde{\mathbf{M}}_i$ is defined as the extended mass of *composite body* i that consisting of rigidly connected links, $\#i, \cdots, \#n$, with respect to the mass center of the ith link, C_i.

Now, eq.(9b) is substituted into the expression for $\mathbf{I}$, eq.(13), which leads to the desired expression for the GIM, i.e.,

$$\mathbf{I} \equiv \begin{bmatrix} i_{11} & & \text{sym} \\ \vdots & \ddots & \\ i_{n1} & \cdots & i_{nn} \end{bmatrix}, \text{ where } i_{ij} \equiv \mathbf{p}_i^T\tilde{\mathbf{M}}_i\mathbf{B}_{ij}\mathbf{p}_j \tag{16}$$

for $i = 1, \cdots, j$; $j = 1, \cdots, n$. The word, "sym," denotes the symmetric elements of matrix $\mathbf{I}$.

4 The UDU$^{\mathrm{T}}$ Decomposition

The steps to find the desired decomposition are given below:

1. Based on eqs.(30) and (31), the reverse Gaussian elimination (RGE) is performed on the GIM, $\mathbf{I}$, eq.(16), while $k = n, \cdots, 2$, i.e.,

$$\mathbf{EI} = \mathbf{L}_2, \quad \text{where} \quad \mathbf{E} \equiv \mathbf{E}_2 \cdots \mathbf{E}_n \tag{17}$$

   The matrices, $\mathbf{E}$ and $\mathbf{L}_2$, are, respectively, being the $n \times n$ upper and lower triagular matrices.

2. An essential property of the EUTM, as introduced in eqs.(26)–(29), is stated here as

$$\mathbf{E}_k^{-1} \equiv (\mathbf{1} - \boldsymbol{\alpha}_k \boldsymbol{\lambda}_k^T)^{-1} = \mathbf{1} + \boldsymbol{\alpha}_k \boldsymbol{\lambda}_k^T \tag{18}$$

   where $\boldsymbol{\alpha}_k$ and $\boldsymbol{\lambda}_k$ are defined in eqs.(27) and (28), respectively. Using eq.(18), the GIM, $\mathbf{I}$, is written from eq.(17) as

$$\mathbf{I} = \mathbf{UL}_2, \qquad \text{where} \qquad \mathbf{U} \equiv \mathbf{E}^{-1} \tag{19}$$

   In eq.(19), matrices $\mathbf{U}$ and $\mathbf{L}_2$ are the $n \times n$ upper and lower triangular matrices, respectively. From the inverse of the EUTM, eq.(18), it is clear that the diagonal elements of $\mathbf{U}$ are unity and the above-diagonal elements are the components of vectors, $\boldsymbol{\alpha}_k$, for $k = 2, \cdots, n$, which are evaluated from eq.(31).

3. Since the decomposition of $\mathbf{I}$ given by eq.(19) is not unique [6], a unique decomposition is obtained by normalizing the elements of $\mathbf{L}_2$ as

$$\mathbf{L}_2 = \mathbf{DL}, \text{ where } \mathbf{D} \equiv \mathrm{diag}[\hat{m}_1, \; \cdots, \; \hat{m}_n] \tag{20}$$

   $\mathbf{D}$ being the $n \times n$ matrix whose non zero diagonal elements are those of matrix $\mathbf{L}_2$, as calculated using eq.(32). Hence, the diagonal elements of matrix $\mathbf{L}$ are unity.

4. Finally, for the symmetric matrix, $\mathbf{I}$, $\mathbf{L} \equiv \mathbf{U}^T$ [6]. Therefore, the desired decomposition of the manipulator GIM, $\mathbf{I}$, eq.(16), is

$$\mathbf{I} = \mathbf{UDU}^T \tag{21}$$

   where the elements of the matrices, $\mathbf{U}$ and $\mathbf{D}$, are evaluated using eqs.(31)–(35).

Note also that the matrix, $\mathbf{U}$, can be decomposed as

$$\mathbf{U} = \mathbf{1} + \mathbf{T}_d^T (\mathbf{T}_l - \mathbf{1})^T \mathbf{N} \mathbf{T}_d \tag{22a}$$

where $\mathbf{1}$ is the $n \times n$ identity matrix and matrices $\mathbf{T}_l$ and $\mathbf{T}_d$ are defined in eqs.(9a) and (9b), respectively, whereas the $6n \times 6n$ matrix, $\mathbf{N}$, is as follows:

$$\mathbf{N} \equiv \mathrm{diag}[\frac{\hat{\mathbf{M}}_{12}}{\hat{m}_1}, \; \cdots, \; \frac{\hat{\mathbf{M}}_{n,n+1}}{\hat{m}_n}] \tag{22b}$$

in which $\hat{\mathbf{M}}_{i,i+1}$, for $i = n, \cdots, 1$, is obtained from eq.(35).

The decomposition of matrix $\mathbf{U}$, eq.(22a), is similar to that of [5], which is based on the Kalman filtering. Moreover, contrary to the definition of the extended mass of *composite body i*, $\tilde{\mathbf{M}}_i$, matrix $\hat{\mathbf{M}}_{i,i+1}$ implies the extended mass of *articulated body i*—a system comprising of links $\#i, \cdots, \#n$ that are coupled by joints $i+1, \cdots, n$, as shown in Fig. 2—with respect to the mass center of the ith link, C_i. Thus, matrix $\hat{\mathbf{M}}_{i,i+1}$ is the *articulated-body* inertia of link i, as termed in [4], and, referred to [5], as the *state estimation error covariance*, which satisfies the discrete Riccati equations. The scalar, $\hat{m}_i$, is, however, interpreted in this paper as the moment of inertia of *articulated body i* about the axis of rotation of the ith revolute joint.

5 Forward Dynamics

An application of the proposed decomposition of the manipulator inertia matrix is the development of an $\mathcal{O}(n)$ recursive algorithm for forward dynamics, where the joint accelerations are solved from the dynamic equations of motion. Let the equations of motion of the n-link n-degrees of freedom (DOF) manipulator, as shown in Fig. 2, is given by

$$\mathbf{I}\ddot{\boldsymbol{\theta}} = \boldsymbol{\tau} \tag{23}$$

where $\mathbf{I}$ is the GIM, eq.(16), and $\ddot{\boldsymbol{\theta}}$ is the n-dimensional vector of joint accelerations, the time derivative of $\dot{\boldsymbol{\theta}}$, as defined in eq.(9c), whereas $\boldsymbol{\tau}$ denotes the n-dimensional vector of joint torques or forces due to known external applied moments and forces, and those, resulting from the gravity, centrifugal and coriolis accelerations. Vector $\boldsymbol{\tau}$ is assumed to be efficiently calculated from an $\mathcal{O}(n)$ inverse dynamics algorithm, e.g., [7], while $\ddot{\boldsymbol{\theta}} = \mathbf{0}$, and the known moments and forces. Thus, in order to render an $\mathcal{O}(n)$ forward dynamics algorithm, it is necessary to solve $\ddot{\boldsymbol{\theta}}$ from eq.(23) with $\mathcal{O}(n)$ computations. This is done by solving the following three sets of equations, which are obtained by substituting eq.(21) into eq.(23), i.e.,

$$\mathbf{U}\hat{\boldsymbol{\tau}} = \boldsymbol{\tau}, \qquad \mathbf{D}\tilde{\boldsymbol{\tau}} = \hat{\boldsymbol{\tau}}, \qquad \text{and} \qquad \mathbf{U}^T\ddot{\boldsymbol{\theta}} = \tilde{\boldsymbol{\tau}} \tag{24}$$

The recursive schemes to obtain $\ddot{\boldsymbol{\theta}}$ are shown in §**B**.

6 Conclusions

Based on the knowledge of elementary Linear Algebra, the symbolic $\mathbf{UDU}^T$ decomposition of the generalized inertia matrix (GIM) of a serial manipulator is proposed, where $\mathbf{U}$ and $\mathbf{D}$ are the unit upper triangular and diagonal matrices, respectively. The symbolic operations were possible due to the representation of the elements of the GIM as expressions that owes to the decoupling of the natural orthogonal complement (NOC), as done eq.(8). Since the decomposition shows recursive relations among the elements of the associated matrices, an $\mathcal{O}(n)$ forward dynamics algorithm could be developed. Moreover, several physical interpretations are provided, as after eqs.(15c) and (22b). Without the decoupled NOC, as in [8, 9], the GIM can only be calculated using an algorithm, resulting in a numerical representation. This calls for numerical decomposition, which prohibits to recognize the recursive relations.

The proposed $\mathcal{O}(n)$ forward dynamics scheme requires $(201n - 335)$ multiplications or divisions and $(193n - 361)$ additions or subtractions, which, compared to the $\mathcal{O}(n^3)$ algorithm [7], is not efficient while $n < 12$. Thus, like the forward dynamics scheme in [4], the algorithm is only appreciable for those manipulators having redundancy of six or more.

The $\mathbf{UDU}^T$ decomposition is, nevertheless, useful for analytical investigation of the manipulator dynamics. For example, the effects of individual link parameters on the composite bodies of the system can be evaluated from eq.(15a), which can help a robot designer. Also, the recursive equations can be used to verify the intermediate steps of forward dynamics calculations. The similar findings are also reported in [4] and [5], which are based on the definition of the *articulated-body* inertia and the Kalman filtering, respectively. The $\mathbf{UDU}^T$ decomposition, on the other hand, uses simple rules of the Gaussian elimination, which are easy to understand and follow.

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A. Reverse Gaussian Elimination

Conventionally, the Gaussian elimination (GE) [6] begins from the first column of the matrix under interest. In the proposed elimination, however, it is assumed that the GE of matrix $\mathbf{I}$, eq.(16), starts from the nth column. Thus, the name *reverse* Gaussian elimination (RGE) is used. In the RGE, after the annihilation of the first $(n-1)$ elements of the nth column, the modified inertia matrix, denoted by $\mathbf{L}_n$, is given as

$$\mathbf{L}_n \equiv \begin{bmatrix} i_{11}^{(n)} & & \text{sym} & 0 \\ \vdots & \ddots & & 0 \\ i_{n-1,1}^{(n)} & \cdots & i_{n-1,n-1}^{(n)} & 0 \\ i_{n1} & \cdots & i_{n,n-1} & i_{nn} \end{bmatrix} \quad (25)$$

where i_{nn} is the *pivot* [6] and $i_{ij}^{(n)}$ are the modified elements of $\mathbf{I}$, whereas "sym" denotes the symmetric elements of the $(n-1)\times(n-1)$ matrix, resulting from the deletion of the nth row and column of matrix $\mathbf{L}_n$. Equation (25) is realized by premultiplying matrix $\mathbf{I}$ with the *elementary upper triangular matrix* (EUTM) of order n and index n, as done in GE with *elementary lower triangular matrix* (ELTM) [6]. An EUTM of order n and index k, denoted by $\mathbf{E}_k$, is defined similar to an ELTM as

$$\mathbf{E}_k \equiv \mathbf{1} - \boldsymbol{\alpha}_k \boldsymbol{\lambda}_k^T \tag{26}$$

where $\mathbf{1}$ is the $n \times n$ identity matrix and the n-dimensional vectors, $\boldsymbol{\alpha}_k$ and $\boldsymbol{\lambda}_k$, are defined by

$$\boldsymbol{\alpha}_k \equiv [\alpha_{1k},\ \cdots,\ \alpha_{k-1,k},\ 0,\ \cdots,\ 0]^T \tag{27}$$

$$\boldsymbol{\lambda}_k \equiv [\ 0,\ \ \cdots,\ \ 0,\ \ 1,\ \cdots,\ 0]^T \tag{28}$$

From eqs.(27) and (28), the EUTM, $\mathbf{E}_k$, eq.(26), has the following structure:

$$\mathbf{E}_k \equiv \begin{bmatrix} 1 & 0 & \cdots & -\alpha_{1k} & \cdots & 0 \\ & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & 1 & -\alpha_{k-1,k} & \cdots & 0 \\ & & & 1 & \cdots & 0 \\ & & 0's & & \ddots & \vdots \\ & & & & & 1 \end{bmatrix} \tag{29}$$

where "$0's$" imply zeros. Moreover, in the RGE, the modified matrix after the annihilation of the first $k-1$ elements of the kth column is expressed as

$$\mathbf{L}_k = \mathbf{E}_k \mathbf{L}_{k+1} \tag{30}$$

where, if $k = n$ and $\mathbf{L}_{n+1} \equiv \mathbf{I}$, the matrix, $\mathbf{L}_n$ of eq.(25), immediately follows. Furthermore, the elements of $\mathbf{E}_k$ and $\mathbf{L}_k$, α_{ik} and $i_{ij}^{(k)}$, respectively, are computed from the following scheme, i.e.,

- For $k = n, \cdots, 2$; Do $i = k-1, \cdots, 1$; Do $j = i, \cdots, 1$

  $$\alpha_{ik} = \mathbf{p}_i^T \hat{\boldsymbol{\psi}}_{ik} \quad \text{and} \quad i_{ij}^{(k)} = \mathbf{p}_i^T \hat{\mathbf{M}}_{ik} \mathbf{B}_{ij} \mathbf{p}_j \tag{31}$$

  end do j; end do i; end for k.

In eq.(31), matrix $\mathbf{B}_{ij}$, and vectors $\mathbf{p}_i$ and $\mathbf{p}_j$ are defined in eq.(4), whereas the 6-dimensional vector $\hat{\boldsymbol{\psi}}_{ik}$ and the terms associated to it are written as

$$\boldsymbol{\psi}_k \equiv \hat{\mathbf{M}}_{k,k+1}\mathbf{p}_k;\ \boldsymbol{\psi}_{ik} \equiv \mathbf{B}_{ki}^T \boldsymbol{\psi}_k;\ \hat{m}_k \equiv \mathbf{p}_k^T \boldsymbol{\psi}_k \tag{32}$$

$$\hat{\boldsymbol{\psi}}_k \equiv \frac{\boldsymbol{\psi}_k}{\hat{m}_k}; \qquad \hat{\boldsymbol{\psi}}_{ik} \equiv \frac{\boldsymbol{\psi}_{ik}}{\hat{m}_k} \tag{33}$$

where the 6×6 matrix, $\hat{\mathbf{M}}_{ik}$, eq.(31), or $\hat{\mathbf{M}}_{k,k+1}$, eq.(32), is evaluated from the following relation:

$$\hat{\mathbf{M}}_{ik} \equiv \tilde{\mathbf{M}}_i - \tilde{\boldsymbol{\Psi}}_{ik}, \quad \text{where} \quad \tilde{\boldsymbol{\Psi}}_{ik} \equiv \sum_{l=k}^{n} \hat{\boldsymbol{\psi}}_{il} \boldsymbol{\psi}_{il}^T \tag{34}$$

in which $\tilde{\mathbf{M}}_i$ is given in eq.(15a). Note that the 6×6 matrix, $\tilde{\boldsymbol{\Psi}}_{ik}$, has a recursive relation, whose substitution, along with that of $\tilde{\mathbf{M}}_{i,i+1}$, eq.(15c), into eq.(34), leads to a recursive relation for $\hat{\mathbf{M}}_{ik}$, i.e., for $k = n, \cdots, 2$; $i = k-1, \cdots, 1$

$$\hat{\mathbf{M}}_{ik} = \mathbf{M}_i + \mathbf{B}_{i+1,i}^T \hat{\mathbf{M}}_{i+1,k} \mathbf{B}_{i+1,i} \tag{35}$$

where, if $i = k-1$, $\hat{\mathbf{M}}_{i+1,k} \equiv \hat{\mathbf{M}}_{kk} = \hat{\mathbf{M}}_{k,k+1} - \hat{\boldsymbol{\psi}}_k \boldsymbol{\psi}_k^T$, and $\hat{\mathbf{M}}_{n,n+1} \equiv \mathbf{M}_n$.

B. Joint Accelerations

In order to evaluate the joint accelerations, $\ddot{\boldsymbol{\theta}}$, for input $\boldsymbol{\tau}$, three sets of equations, given by eq.(24), are solved in the following steps:

1. *Solution for* $\hat{\boldsymbol{\tau}}$: The solution, $\hat{\boldsymbol{\tau}} = \mathbf{U}^{-1}\boldsymbol{\tau}$, is evaluated as, $\hat{\boldsymbol{\tau}} = \mathbf{E}_2, \cdots, \mathbf{E}_n\boldsymbol{\tau}$, which has the following recursion: For $i = n-1, \cdots, 1$

   $$\hat{\tau}_i = \tau_i - \mathbf{p}_i^T \tilde{\boldsymbol{\eta}}_{i,i+1} \tag{36}$$

   where $\hat{\tau}_n \equiv \tau_n$, and the 6-dimensional vector, $\tilde{\boldsymbol{\eta}}_{i,i+1}$, is given as

   $$\tilde{\boldsymbol{\eta}}_{i,i+1} \equiv \mathbf{B}_{i+1,i}^T \tilde{\boldsymbol{\eta}}_{i+1} \tag{37}$$

   in eq.(37), $\tilde{\boldsymbol{\eta}}_{i+1} \equiv \hat{\boldsymbol{\psi}}_{i+1}\hat{\tau}_{i+1} + \tilde{\boldsymbol{\eta}}_{i+1,i+2}$; $\tilde{\boldsymbol{\eta}}_{n,n+1} = \mathbf{0}$.

2. *Solution for* $\tilde{\boldsymbol{\tau}}$: The solution of the equation, $\mathbf{D}\tilde{\boldsymbol{\tau}} = \hat{\boldsymbol{\tau}}$, involves the inversion of the diagonal matrix, $\mathbf{D}$ of eq.(20), which is simple, namely, $\mathbf{D}^{-1}$ has only nonzero diagonal elements that are the reciprocal of the corresponding elements of $\mathbf{D}$. Vector $\tilde{\boldsymbol{\tau}}$ is obtained below: For $i = 1, \cdots, n$

   $$\tilde{\tau}_i = \hat{\tau}_i / \hat{m}_i$$

3. *Solution for* $\ddot{\boldsymbol{\theta}}$: In this step, $\ddot{\boldsymbol{\theta}} \equiv \mathbf{U}^{-T}\tilde{\boldsymbol{\tau}}$, is calculated as $\mathbf{E}^T\tilde{\boldsymbol{\tau}}$, i.e., for $i = 2, \cdots, n$

   $$\ddot{\theta}_i = \tilde{\tau}_i - \hat{\boldsymbol{\psi}}_i^T \tilde{\boldsymbol{\mu}}_{i,i-1} \tag{38}$$

   where $\ddot{\theta}_1 \equiv \tilde{\tau}_1$, and the 6-dimensional vector, $\tilde{\boldsymbol{\mu}}_{i,i-1}$, is obtained from

   $$\tilde{\boldsymbol{\mu}}_{i,i-1} \equiv \mathbf{B}_{i,i-1}\tilde{\boldsymbol{\mu}}_{i-1} \tag{39}$$

   in which, $\tilde{\boldsymbol{\mu}}_{i-1} \equiv \mathbf{p}_{i-1}\ddot{\theta}_{i-1} + \tilde{\boldsymbol{\mu}}_{i-1,i-2}$; $\tilde{\boldsymbol{\mu}}_{10} = \mathbf{0}$.