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Analytical Expression for the Inverted Inertia Matrix of Serial Robots

Abstract

This paper presents the analytical derivation of the inertia matrix and its inverse for an open-loop, serial-chain robot. The derivation allows one to write a recursive forward-dynamics algorithm for simulation purposes whose computational complexity is of order n*, i.e.,* $\mathcal{O}(n)$ n *being the degrees of freedom of the robot under study. The proposed methodology is based on the Gaussian elimination of the inertia matrix, in contrast to, say, Kalman filtering, which is proposed elsewhere. The derivation is illustrated with a three-degrees-of-freedom planar robot.*

1. Introduction

The inertia matrix in the dynamic equations of motion plays an important role in simulation, which has two parts, namely, (1) the solution of the joint accelerations from the equations of motion, that is, forward dynamics; and (2) numerical integration of the joint accelerations. Note that the traditional forward-dynamics algorithms are of $\mathcal{O}(n^3)$ (Walker and Orin 1982; Angeles and Ma 1988). This is due to the numerical approach to the problem. In those approaches, for example, in the work of Walker and Orin (1982), first, the elements of the inertia matrix are calculated as numbers; then, the decomposition, say, a Cholesky decomposition (Stewart 1973) of the resultant matrix is performed, Finally the joint accelerations are solved by backward and forward substitutions. Since the complexity of Cholesky decomposition is of the order n^3, that is, $\mathcal{O}(n^3)$, the forward-dynamics algorithm also requires $\mathcal{O}(n^3)$ computations.

Alternatively, recursive forward-dynamics schemes (Armstrong 1979; Featherstone 1983; Rodriguez and Kreutz-Delgado 1992) have $\mathcal{O}(n)$ computational complexities. These algorithms, compared with the $\mathcal{O}(n^3)$ schemes, are efficient only when $n \geq 12$, however, calculate the joint accelerations that are smooth functions of time (Cloutier, Pai, and Asche, 1995). As a result, numerical integration is faster. Hence, when using the $\mathcal{O}(n)$ schemes, the total CPU time for the simulation may be even less for $n < 12$ (Cloutier et al., 1995). Besides, $\mathcal{O}(n)$ algorithms provide a physical interpretation of each step they follow. Thus, the recursive robot dynamics is becoming more popular. Their computational complexities, as given in Section 7, however, vary depending on the approaches. For example, Featherstone (1983) uses a concept what he calls *articulated body inertia*, whereas Rodriguez and Kreutz-Delgado (1992) developed their algorithm based on control theory, namely, Kalman filtering and a smoothing technique, arising in the state estimation.

The proposed approach is based on the analytical Gaussian Elimination (GE)[1] of the inertia matrix (Saha 1997); that is, the GE rules are applied to the analytical expressions of the elements of the inertia matrix. This allows one to write the analytical expressions for the elements of the inverse of the inertia matrix also. Interestingly, the elements of the inverted inertia matrix can be obtained recursively. As a result, a recursive $\mathcal{O}(n)$ forward-dynamics algorithm can be developed (Saha 1997). This paper focuses on the former issue, that is, writing the analytical expressions for the inverted inertia matrix. The following contributions are made in this paper:

- Derivation of the explicit analytical expressions for the elements of the inverted inertia matrix;
- Physical interpretation of the terms obtained during the inversion;
- Finding a correlation between the proposed methodology and that of Rodriguez and Kreutz-Delgado (1992). The comparison of the results, as discussed in Section 5.1, reveals that the Gaussian elimination and the Kalman filtering are equivalent under certain conditions. The real proof is, however, beyond the scope of this paper.

The organization of the paper is as follows: First, some definitions pertaining to the development of the inertia matrix

1. **It can be verified easily that the GE of a symmetric positive- definite matrix is equivalent to the Cholesky decomposition.**

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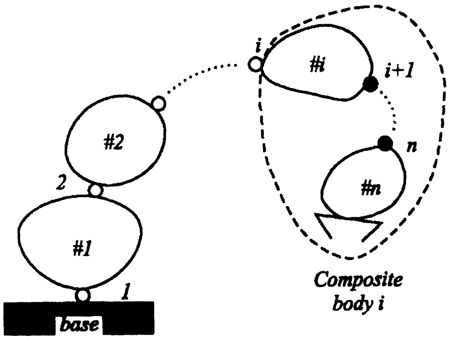

Fig. 1. An n-link n-DOF serial robot.

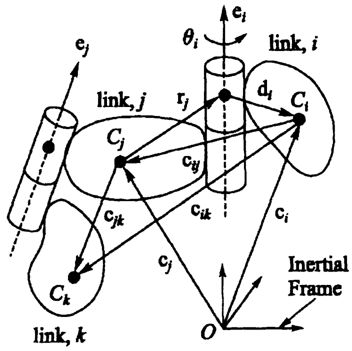

Fig. 2. A system of coupled links.

and its inverse are given in Section 2. Section 3 develops the dynamic modeling using the Decoupled Natural Orthogonal Complement (DeNOC) matrices. In Section 4, the inertia matrix is obtained analytically, followed by its inverse in Section 5. Finally, to illustrate the proposed methodology, a 3-link planar robot is studied in Section 6 before the conclusions are made in Section 7.

2. Some Definitions

Referring to the motion of the ith body, as shown in Figure 1, the following terms are defined:

$\mathbf{t}_i$ and $\mathbf{n}_i$: the 6-dimensional vector of *twist* and *wrench* of the ith link; that is,

$$\mathbf{t}_i \equiv \begin{bmatrix} \boldsymbol{\omega}_i \\ \mathbf{v}_i \end{bmatrix} \quad \text{and} \quad \mathbf{w}_i \equiv \begin{bmatrix} \mathbf{n}_i \\ \mathbf{f}_i \end{bmatrix}, \tag{1}$$

where $\boldsymbol{\omega}_i$ and $\mathbf{v}_i$ are the 3-dimensional vectors of angular velocity and the linear velocity of the mass center of the ith link, C_i, as shown in Figure 2, respectively, whereas $\mathbf{n}_i$ and $\mathbf{f}_i$ are the 3-dimensional vectors of moment about C_i and the force on the ith body, respectively.

$\boldsymbol{\Omega}_i$, $\mathbf{M}_i$: the 6×6 matrices of extended angular velocity and extended mass of the ith body, respectively, namely,

$$\begin{aligned} \boldsymbol{\Omega}_i &\equiv \begin{bmatrix} \boldsymbol{\omega}_i \times \mathbf{1} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \quad \text{and} \quad (2) \\ \mathbf{M}_i &\equiv \begin{bmatrix} \mathbf{I}_i & \mathbf{O} \\ \mathbf{O} & m_i \mathbf{1} \end{bmatrix}, \end{aligned}$$

where $\boldsymbol{\omega}_i \times \mathbf{1}$ is the 3×3 cross-product tensor such that $(\boldsymbol{\omega}_i \times \mathbf{1})\mathbf{x} \equiv \boldsymbol{\omega}_i \times \mathbf{x}$, for any 3-dimensional Cartesian vector, $\mathbf{x}$. Also, $\mathbf{I}_i$ and m_i are the 3×3 inertia tensor about the mass center, C_i, and the mass of the ith body, respectively. The terms $\mathbf{1}$ and $\mathbf{O}$ are the 3×3 identity and zero matrices, respectively. Note that the matrices $\mathbf{1}$ and $\mathbf{O}$, henceforth should be understood as dimensions that are compatible with the dimensions of the matrix expressions where they appear.

$\mathbf{t}$ and $\mathbf{w}$: the $6n$-dimensional vectors of generalized twist and wrench, respectively, that are defined as

$$\mathbf{t} \equiv \left[\mathbf{t}_1^T, \cdots, \mathbf{t}_n^T\right]^T \quad \text{and} \quad \mathbf{w} \equiv \left[\mathbf{w}_1^T, \cdots, \mathbf{w}_n^T\right]^T, \tag{3}$$

where $\mathbf{t}_i$ and $\mathbf{w}_i$, for $i = 1, \cdots, n$, are given in eq. (1).

$\boldsymbol{\Omega}$ and $\mathbf{M}$: the $6n \times 6n$ matrices of generalized extended angular velocity and mass, respectively; that is,

$$\begin{aligned} \boldsymbol{\Omega} &\equiv \operatorname{diag}[\boldsymbol{\Omega}_1, \cdots, \boldsymbol{\Omega}_n] \quad \text{and} \quad (4) \\ \mathbf{M} &\equiv \operatorname{diag}[\mathbf{M}_1, \cdots, \mathbf{M}_n], \end{aligned}$$

where $\boldsymbol{\Omega}_i$ and $\mathbf{M}_i$ for $i = 1, \cdots, n$, are given in eq. (2), respectively.

$\dot{\boldsymbol{\theta}}$: the n-dimensional vector of joint rates, $\dot{\boldsymbol{\theta}}$, that is,

$$\dot{\boldsymbol{\theta}} \equiv \left[\dot{\theta}_1, \cdots, \dot{\theta}_n\right]^T, \tag{5}$$

where θ_i is the displacement of the ith joint. For a revolute pair, θ_i is shown in Figure 2.

3. Dynamic Modeling Using DeNOC

For a serial robot, as shown in Figure 1, the steps to obtain the dynamic equations of motion using the DeNOC matrices are summarized below:

1. Write the Newton-Euler equations of motion for an uncoupled rigid body from the chain, say, #i, as

$$\mathbf{I}_i\dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \mathbf{I}_i\boldsymbol{\omega}_i = \mathbf{n}_i \quad (6a)$$
$$m_i\ddot{\mathbf{c}}_i = \mathbf{f}_i. \quad (6b)$$

The above six-scalar equations can be put in a compact form as

$$\mathbf{M}_i\dot{\mathbf{t}}_i + \boldsymbol{\Omega}_i\mathbf{M}_i\mathbf{t}_i = \mathbf{w}_i, \quad (7)$$

where $\mathbf{t}_i$, $\mathbf{w}_i$ and $\boldsymbol{\Omega}_i$, $\mathbf{M}_i$ are defined in eqs. (1) and (2), respectively. Furthermore, $\dot{\mathbf{t}}_i$ is the time derivative of the twist, $\mathbf{t}_i$.

2. Write eq. (7), for $i = 1, \cdots, n$, where n is the number of moving links in the serial chain of the robot. The $6n$ scalar equations are then written as

$$\mathbf{M}\dot{\mathbf{t}} + \boldsymbol{\Omega}\mathbf{M}\mathbf{t} = \mathbf{w}, \quad (8)$$

where $\dot{\mathbf{t}}$ is the time derivative of the generalized twist, $\mathbf{t}$. The other terms are defined in eqs. (3) and (4).

3. Write the velocity constraints between two rigid bodies, namely, $\#i$ and $\#j$, coupled by a revolute joint, as shown in Fig. 2, as

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_j + \dot{\theta}_i\mathbf{e}_i \quad (9a)$$
$$\mathbf{v}_i = \mathbf{v}_j + \boldsymbol{\omega}_j \times \mathbf{r}_j + \boldsymbol{\omega}_i \times \mathbf{d}_i. \quad (9b)$$

The above six-scalar equation is written in a compact form, similar to eq. (7), as

$$\mathbf{t}_i = \mathbf{B}_{ij}\,\mathbf{t}_j + \mathbf{p}_i\dot{\theta}_i, \quad (10)$$

where the 6×6 matrix, $\mathbf{B}_{ij}$, and the 6-dimensional vector, $\mathbf{p}_i$, are as follows:

$$\mathbf{B}_{ij} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{c}_{ij} \times \mathbf{1} & \mathbf{1} \end{bmatrix} \quad \text{and} \quad (11)$$
$$\mathbf{p}_i \equiv \begin{bmatrix} \mathbf{e}_i \\ \mathbf{e}_i \times \mathbf{d}_i \end{bmatrix},$$

$\mathbf{c}_{ij} \times \mathbf{1}$ being the cross-product tensor associated with the vector $\mathbf{c}_{ij}$, defined similar to $\boldsymbol{\omega}_i \times \mathbf{1}$ of eq. (2). Vector $\mathbf{c}_{ij}$ $(\equiv -\mathbf{r}_j - \mathbf{d}_i)$ is shown in Fig. 2. Moreover, $\mathbf{e}_i$ is the unit vector parallel to the axis of rotation of the ith revolute joint. Furthermore, due to the serial connection of the robot links, the following relations hold:

$$\mathbf{B}_{ij}\mathbf{B}_{jk} = \mathbf{B}_{ik}, \quad \mathbf{B}_{ii} = \mathbf{1}, \quad \text{and} \quad (12)$$
$$\mathbf{B}_{ij}^{-1} = \mathbf{B}_{ji}.$$

It is pointed out here that eqs. (10) and (11) are written for a revolute pair. In the presence of other joints, say, prismatic, eq. (10) remains valid with only one exception: in the expression of $\mathbf{p}_i$, that is,

$$\mathbf{p}_i \equiv \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_i \end{bmatrix} \quad \text{For prismatic joint,} \quad (13)$$

where $\mathbf{e}_i$ is the unit vector parallel to the axis of linear motion. Correspondingly, $\dot{\theta}_i$ of eq. (10) represents the linear joint rate. Other joints are not treated here, because these two joints are commonly used in serial robots.

4. Write eq. (10) for $i = 1, \cdots, n$. The resultant $6n$ scalar velocity constraints are then obtained as

$$\mathbf{t} = \mathbf{N}\dot{\boldsymbol{\theta}}, \quad \text{where} \quad \mathbf{N} \equiv \mathbf{N}_l\mathbf{N}_d. \quad (14)$$

The $6n \times n$ matrix, $\mathbf{N}$, is the Natural Orthogonal Complement (NOC), introduced by Angeles and Lee (1988), and its decoupled form, $\mathbf{N} \equiv \mathbf{N}_l\mathbf{N}_d$, is termed as the *Decoupled* NOC (DeNOC) (Saha 1995). The latter provides the recursive relations for the elements of both the inertia matrix and its inverse, as derived in Sections 4 and 5, respectively. The $6n \times 6n$ matrix, $\mathbf{N}_l$, and the $6n \times 6n$ matrix, $\mathbf{N}_d$, are written using eqs. (10) and (12), as

$$\mathbf{N}_l \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{B}_{21} & \mathbf{1} & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{n1} & \mathbf{B}_{n2} & \cdots & \mathbf{1} \end{bmatrix} \quad \text{and} \quad (15)$$
$$\mathbf{N}_d \equiv \begin{bmatrix} \mathbf{p}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{p}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{p}_n \end{bmatrix},$$

$\mathbf{O}$ and $\mathbf{0}$ being 6×6 matrix and the 6-dimensional vector of zeros, respectively.

5. Premultiply $\mathbf{N}^T$ with eq. (8): that is,

$$\mathbf{N}^T(\mathbf{M}\dot{\mathbf{t}} + \boldsymbol{\Omega}\mathbf{M}\mathbf{t}) = \mathbf{N}^T(\mathbf{w}^E + \mathbf{w}^I), \quad (16)$$

where $\mathbf{w}$ is substituted as $\mathbf{w} \equiv \mathbf{w}^E + \mathbf{w}^I$; $\mathbf{w}^E$ and $\mathbf{w}^I$ are the generalized external and internal constraint wrenches, respectively. Since constraint moments and forces do not do any work, it is obvious that for independent $\dot{\boldsymbol{\theta}}$, $\mathbf{N}^T\mathbf{w}^I = \mathbf{0}$. Thus, for the n-DOF serial robot under study, as shown in Figure 1, eq. (16) is rewritten as

$$\mathbf{N}^T(\mathbf{M}\dot{\mathbf{t}} + \boldsymbol{\Omega}\mathbf{M}\mathbf{t}) = \mathbf{N}^T\mathbf{w}^E. \quad (17)$$

6. Substitute $\mathbf{t}$, eq. (14), and its time derivative, $\dot{\mathbf{t}} = \mathbf{N}\ddot{\boldsymbol{\theta}} + \dot{\mathbf{N}}\dot{\boldsymbol{\theta}}$, into eq. (17). The result is the following constrained dynamic equations of motion:

$$\mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}} = \boldsymbol{\tau}, \quad (18)$$

where

$\mathbf{I} \equiv \mathbf{N}^T\mathbf{M}\mathbf{N}$: the $n \times n$ inertia matrix;

$\mathbf{C} \equiv \mathbf{N}^T(\mathbf{M}\dot{\mathbf{N}} + \mathbf{\Omega}\mathbf{M}\mathbf{N})$: the $n \times n$ matrix of convective inertia terms;

$\boldsymbol{\tau} \equiv \mathbf{N}^T\mathbf{W}^E$: the n-dimensional vector of generalized forces due to driving torques and those resulting from gravity and dissipation.

4. Analytical Expression for the Inertia Matrix

The expressions for the elements of the inertia matrix, $\mathbf{I}$ of eq. (18), are derived in this section. They play an important role in forward dynamics where the joint accelerations, $\ddot{\boldsymbol{\theta}}$, are calculated as $\ddot{\boldsymbol{\theta}} = \mathbf{I}^{-1}(\boldsymbol{\tau} - \mathbf{C}\dot{\boldsymbol{\theta}})$. In the present discussion, $\boldsymbol{\tau} - \mathbf{C}\dot{\boldsymbol{\theta}}$ is assumed to be input, which can be calculated using a recursive inverse-dynamics algorithm as suggested by Walker and Orin (1982), with $\mathcal{O}(n)$ computations. Note that the use of the NOC matrix in its original form (Angeles and Lee 1988) does not give recursive expressions for the elements of the inertia matrix. The property is, however, apparent while the decoupled form of the NOC, that is, $\mathbf{N}_l\mathbf{N}_d$ is used. It prompted to apply the decomposition rules on the resulting elements of the inertia matrix, expecting recursive relations among the elements of the decomposed matrices. The expectations turned out to be true. Hence a recursive forward-dynamics algorithm could be developed (Saha 1997).

The derivation is as follows. Using the DeNOC matrices, namely, $\mathbf{N}_l$ and $\mathbf{N}_d$, the inertia matrix $\mathbf{I}$ as defined after eq. (18) is expressed as

$$\mathbf{I} = \mathbf{N}_d^T\tilde{\mathbf{M}}\mathbf{N}_d, \quad \text{where} \quad \tilde{\mathbf{M}} \equiv \mathbf{N}_l^T\mathbf{M}\mathbf{N}_l \tag{19}$$

in which the $6n \times 6n$ matrix, $\mathbf{N}_l$, and the $6n \times 6n$ matrix, $\mathbf{N}_d$, are already defined in eq. (15).
The $6n \times 6n$ symmetric matrix, $\tilde{\mathbf{M}}$ (eq. (19)), is given here as

$$\tilde{\mathbf{M}} \equiv \begin{bmatrix} \tilde{\mathbf{M}}_1 & \mathbf{B}_{21}^T\tilde{\mathbf{M}}_2 & \cdots & \mathbf{B}_{n1}^T\tilde{\mathbf{M}}_n \\ \tilde{\mathbf{M}}_2\mathbf{B}_{21} & \tilde{\mathbf{M}}_2 & \cdots & \mathbf{B}_{n2}^T\tilde{\mathbf{M}}_n \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathbf{M}}_n\mathbf{B}_{n1} & \tilde{\mathbf{M}}_n\mathbf{B}_{n2} & \cdots & \tilde{\mathbf{M}}_n \end{bmatrix}, \tag{20}$$

where the 6×6 matrix, $\tilde{\mathbf{M}}_i$, for $i = n, \cdots, 1$, is evaluated recursively as

$$\tilde{\mathbf{M}}_i = \mathbf{M}_i + \mathbf{B}_{i+1,i}^T\tilde{\mathbf{M}}_{i+1}\mathbf{B}_{i+1,i} \tag{21}$$

in which $\tilde{\mathbf{M}}_{n+1} = \mathbf{O}$, because there is no $(n+1)$st link in the kinematic chain. Hence, $\tilde{\mathbf{M}}_n = \mathbf{M}_n$. Matrix $\tilde{\mathbf{M}}_i$ is interpreted as the extended mass of the *composite body* consisting of the rigidly connected bodies, $\#i, \cdots, \#n$ (Fig. 1). Finally, the inertia matrix $\mathbf{I}$, as in eq. (19), is expressed as

$$\mathbf{I} = \begin{bmatrix} i_{11} & & \text{sym} \\ \vdots & \ddots & \\ i_{n1} & \cdots & i_{nn} \end{bmatrix}, \quad \text{where} \tag{22}$$

$$i_{ij} \equiv \mathbf{p}_i^T\tilde{\mathbf{M}}_i\mathbf{B}_{ij}\mathbf{p}_j$$

for $i = 1, \cdots, n; j = 1, \cdots, i$. The word "sym" denotes the symmetric elements of matrix $\mathbf{I}$.

5. Analytical Derivation of the Inverted Inertia Matrix

The inverse of the inertia matrix is required in forward dynamics, as indicated in the previous section. It is true that the explicit inversion is not required, as the joint accelerations can be solved by decomposing the inertia matrix, followed by backward and forward substitutions. However, an explicit evaluation of the inertia matrix, if possible, is advantageous, mainly for debugging and further investigation of the multibody dynamics problem.

In this section, the explicit inversion is done in two steps. First, a decomposition of the inertia matrix is done using the *reverse* Gaussian elimination scheme. Next, the analytical expressions for the elements of the inverted inertia are found by inverting the elementary matrices arising from the decomposition. The inverse is also obtained by Rodriguez and Kreutz-Delgado (1992) using Kalman filtering and a smoothing technique.

5.1. Reverse Gaussian Elimination

In contrast to the conventional Gaussian elimination process (Stewart 1973), where elimination starts from the first column to make all its elements except the top zero, the proposed elimination begins with the last column, where all elements except the bottom are made zero. Thus the name *reverse* Gaussian elimination is used. The steps are as follows:

1. Define an *elementary upper triangular matrix* (EUTM), similar to the *elementary lower triangular matrix* used in conventional Gaussian elimination (Stewart 1973) to annihilate all elements above the *pivot*, as illustrated in the appendix for the nth column. An EUTM of order n and index k, $\mathbf{E}_k$, is defined by

$$\mathbf{E}_k \equiv \mathbf{1} - \boldsymbol{\alpha}_k\boldsymbol{\lambda}_k^T, \tag{23}$$

where $\mathbf{1}$ is the $n \times n$ identity matrix, and the n-dimensional vectors, $\boldsymbol{\alpha}_k$ and $\boldsymbol{\lambda}_k$, are given by

$$\boldsymbol{\alpha}_k \equiv [\alpha_{1k}, \cdots, \alpha_{k-1,k}, 0, \cdots, 0]^T \tag{24}$$

$$\boldsymbol{\lambda}_k \equiv [\ 0, \ \cdots, \ 0, \quad 1, \cdots, 0]^T. \tag{25}$$

The matrix representation of eq. (23) is shown in eq. (51) for $k = n$.

2. Premultiply $\mathbf{I}$, eq. (22), by $\mathbf{E}_k$, for $k = n, \cdots, 2$, which results in an $n \times n$ lower triangular matrix, $\mathbf{L}$, that is,

$$\mathbf{E}_2 \cdots \mathbf{E}_n \mathbf{I} = \mathbf{L}. \qquad (26)$$

Thus, the matrix, $\mathbf{I}$, has the following representation:

$$\mathbf{I} = \mathbf{E}_n^{-1} \cdots \mathbf{E}_2^{-1} \mathbf{L}, \qquad (27)$$

where $\mathbf{E}_i^{-1}$ for $i = 2, \cdots, n$ are the inverse of the EUTMs.

3. Note the inverse of an EUTM, $\mathbf{E}_k$, eq. (23), is

$$\mathbf{E}_k^{-1} \equiv (\mathbf{1} - \alpha_k \boldsymbol{\lambda}_k^T)^{-1} = \mathbf{1} + \alpha_k \boldsymbol{\lambda}_k^T, \qquad (28)$$

which is an $n \times n$ unit upper triangular matrix.

4. The multiplication of the inverses of two neighboring EUTMs, namely, $\mathbf{E}_k^{-1}$ and $\mathbf{E}_{k-1}^{-1}$, is calculated as

$$\begin{aligned} \mathbf{E}_k^{-1}\mathbf{E}_{k-1}^{-1} &= (\mathbf{1} + \alpha_k \boldsymbol{\lambda}_k^T)(\mathbf{1} + \alpha_{k-1} \boldsymbol{\lambda}_{k-1}^T) \qquad (29) \\ &= \mathbf{1} + \alpha_k \boldsymbol{\lambda}_k^T + \alpha_{k-1} \boldsymbol{\lambda}_{k-1}^T, \end{aligned}$$

where $\boldsymbol{\lambda}_k^T \alpha_{k-1} = 0$. Thus the product $\mathbf{E}_n^{-1}, \cdots, \mathbf{E}_2^{-1}$, is obtained as

$$\begin{aligned} \mathbf{E}_n^{-1} \cdots \mathbf{E}_2^{-1} &= \mathbf{1} + \alpha_n \boldsymbol{\lambda}_n^T + \cdots + \alpha_2 \boldsymbol{\lambda}_2^T \qquad (30) \\ &\equiv \mathbf{U}, \end{aligned}$$

which is a unit upper triangular matrix; that is, the diagonal elements of $\mathbf{U}$ are one.

5. For the symmetric positive-definite matrix, $\mathbf{I}$, it can be shown that the reduced matrix $\mathbf{L}$ of eq. (26) has the following decomposition (Stewart 1973):

$$\mathbf{L} = \mathbf{D}\mathbf{U}^T, \qquad (31)$$

where $\mathbf{D}$ is the $n \times n$ diagonal matrix whose elements are the diagonal elements of matrix $\mathbf{L}$.

6. Hence, a decomposition of the inertia matrix is obtained as

$$\mathbf{I} = \mathbf{U}\mathbf{D}\mathbf{U}^T, \qquad (32)$$

where the $n \times n$ matrices $\mathbf{U}$ and $\mathbf{D}$ are displayed below:

$$\mathbf{U} \equiv \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ & 1 & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & 0's & & 1 \end{bmatrix} \quad \text{and}$$

$$\mathbf{D} \equiv \begin{bmatrix} \hat{m}_1 & & 0's & \\ & \hat{m}_2 & & \\ & & \ddots & \\ & 0's & & \hat{m}_n \end{bmatrix} \qquad (33)$$

in which "0's" implies zeros, and the elements $\hat{m}_i$ and a_{ij} for $i = 1, \cdots, n$, and $j = i, \cdots, n$, are as follows:

$$\begin{aligned} \hat{m}_i &= \mathbf{p}_i^T \hat{\mathbf{M}}_i \mathbf{p}_i \quad \text{and} \qquad (34) \\ a_{ij} &= \frac{1}{\hat{m}_j} \mathbf{p}_i^T \mathbf{B}_{ji}^T \hat{\mathbf{M}}_j \mathbf{p}_j . \end{aligned}$$

The 6×6 matrix, $\hat{\mathbf{M}}_i$, for $i = n, \cdots, 1$, is evaluated recursively as

$$\hat{\mathbf{M}}_i = \mathbf{M}_i + \mathbf{B}_{i+1,i}^T \boldsymbol{\Psi}_{i+1} \hat{\mathbf{M}}_{i+1} \mathbf{B}_{i+1,i}, \qquad (35)$$

where

$$\boldsymbol{\Psi}_{i+1} \equiv \mathbf{1} - \frac{1}{\hat{m}_{i+1}} \hat{\mathbf{M}}_{i+1} \mathbf{p}_{i+1} \mathbf{p}_{i+1}^T$$

and $\hat{\mathbf{M}}_n \equiv \mathbf{M}_n$, because $\hat{\mathbf{M}}_{n+1} \equiv \mathbf{O}$ due to the nonexistence of an $(n+1)$st body in the serial chain.

Matrix $\boldsymbol{\Psi}_{i+1}$ is the 6×6 matrix, which is also derived in Lilly and Orin (1990) using the *inertia propagation method* and is termed the *articulation transformation* matrix. Moreover, matrix $\hat{\mathbf{M}}_i$ has the following interpretation: it is the extended mass of the *articulated body*, *i*. An *i*th articulated body is a system of $(n - i + 1)$ bodies, namely, $\#i, \cdots, \#n$, coupled by $(n - i)$ joints, that is, $i + 1, \cdots, n$. While $\tilde{\mathbf{M}}_i$ contains the inertial parameters of the *rigid* composite body *i*, $\hat{\mathbf{M}}_i$ accounts for the presence of the joints in the rigid composite body. In fact, comparison of the expressions for $\tilde{\mathbf{M}}_i$ and $\hat{\mathbf{M}}_i$, as in eqs. (21) and (35), respectively, shows that the 6×6 matrix $\boldsymbol{\Psi}_{i+1}$ causes this change. This could have prompted Lilly and Orin (1990) to name the matrix as articulation transformation matrix. Matrix $\hat{\mathbf{M}}_i$ is also the *state-estimation error covariance* of Rodriguez and Kreutz-Delgado (1992). Finally, the scalar $\hat{m}_i$, is interpreted as the moment of inertia of the articulated body, *i*, about the axis of rotation of the *i*th joint.

5.2. The Inverted Inertia Matrix

Based on the reverse Gaussian elimination of the inertia matrix, as explained in Section 5.1, the inverse of the inertia matrix, $\mathbf{I}^{-1}$, can be obtained from eq. (32) as

$$\mathbf{I}^{-1} = \mathbf{U}^{-T}\mathbf{D}^{-1}\mathbf{U}^{-1}, \quad \text{where} \quad \mathbf{U}^{-1} \equiv \mathbf{E}_2, \cdots, \mathbf{E}_n . \qquad (36)$$

In eq. (36), $\mathbf{D}^{-1}$ is simply a diagonal matrix whose nonzero diagonal elements are $1/\hat{m}_i$, for $i = 1, \cdots, n$, whereas the elements of $\mathbf{U}^{-1}$ are obtained from the following steps:

1. Using the definition of the EUTM, $\mathbf{E}_k$ of eq. (23), $\mathbf{E}_{k-1}\mathbf{E}_k$ is evaluated as

$$\begin{aligned} \mathbf{E}_{k-1}\mathbf{E}_k &= (\mathbf{1} - \alpha_{k-1} \boldsymbol{\lambda}_{k-1}^T)(\mathbf{1} - \alpha_k \boldsymbol{\lambda}_k^T) \qquad (37) \\ &= \mathbf{1} - \alpha_{k-1} \boldsymbol{\lambda}_{k-1}^T - (\alpha_k - a_{k-1,k}\, \alpha_{k-1}) \boldsymbol{\lambda}_k^T, \end{aligned}$$

where $\alpha_{k-1,k}$ is the last nonzero element of the n-dimensional vector, $\boldsymbol{\alpha}_k$, as in eq. (24). The product, $\mathbf{U}^{-1} \equiv \mathbf{E}_2, \cdots, \mathbf{E}_n$, is then calculated as

$$\mathbf{U}^{-1} \equiv \mathbf{E}_2 \cdots \mathbf{E}_n = \mathbf{1} + \boldsymbol{\alpha}_2^{'}\boldsymbol{\lambda}_2^T + \boldsymbol{\alpha}_3^{'}\boldsymbol{\lambda}_3^T + \cdots + \boldsymbol{\alpha}_n^{'}\boldsymbol{\lambda}_n^T, \qquad (38)$$

in which $\boldsymbol{\alpha}_i^{'}$, for $i = 2, \cdots, n$, is given by

$$\boldsymbol{\alpha}_i^{'} \equiv -(\boldsymbol{\alpha}_i - \alpha_{i-1,i}\boldsymbol{\alpha}_{i-1}^{'} - \cdots - \alpha_{2i}\boldsymbol{\alpha}_2^{'}). \qquad (39)$$

Note above that $\boldsymbol{\alpha}_2^{'} \equiv \boldsymbol{\alpha}_2$.

2. Matrix representation of the matrices, $\mathbf{U}^{-1}$ and $\mathbf{D}^{-1}$, are as follows:

$$\mathbf{U}^{-1} \equiv \begin{bmatrix} 1 & \alpha_{12}^{'} & \cdots & \alpha_{1n}^{'} \\ & 1 & \cdots & \alpha_{2n}^{'} \\ & & \ddots & \vdots \\ & 0's & & 1 \end{bmatrix} \text{ and } \qquad (40)$$

$$\mathbf{D}^{-1} \equiv \begin{bmatrix} \frac{1}{\tilde{m}_1} & & 0's & \\ & \frac{1}{\tilde{m}_2} & & \\ & & \ddots & \\ & 0's & & \frac{1}{\tilde{m}_n} \end{bmatrix},$$

in which $\alpha_{ij}^{'}$, for $i = 1, \cdots, n; j = 1, \ldots, i$, are obtained similarly to α_{ij}, as

$$\alpha_{ij}^{'} = \frac{1}{\hat{m}_j}\mathbf{p}_i^T \hat{\mathbf{B}}_{ji}^T \hat{\mathbf{M}}_j \mathbf{p}_j, \text{ where } \qquad (41)$$

$$\hat{\mathbf{B}}_{ji} \equiv \mathbf{B}_{i,i-1} \prod_{k=i-1}^{j+1} \boldsymbol{\Psi}_k^T \mathbf{B}_{k,k-1}.$$

6. An Example: A 3-Link Planar Robot

To illustrate the methodology of the derivation of the inertia matrix and its inertia, as explained in Sections 4 and 5, consider a 3-link, three-degrees-of-freedom planar robot moving in the $X - Y$ plane, as shown in Figure 3. Let $\mathbf{i}$ and $\mathbf{j}$ be two unit vectors parallel to the axes X and Y, respectively, and let $\mathbf{k} = \mathbf{i} \times \mathbf{j}$.

6.1. The 3 × 3 Inertia Matrix

The inertia matrix of the 3-link, 3-DOF robot moving in the $X - Y$ plane, as shown in Figure 3 is given by

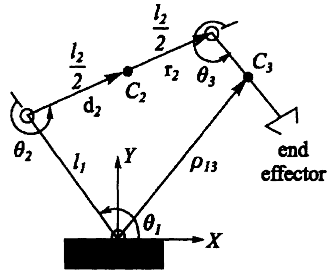

Fig. 3. A 3-link planar robot.

$$\mathbf{I} \equiv \begin{bmatrix} i_{11} & & \text{sym} \\ i_{21} & i_{22} & \\ i_{31} & i_{32} & i_{33} \end{bmatrix}, \qquad (42)$$

where "sym" denotes the symmetric elements of $\mathbf{I}$, and i_{ij}, for $i = 1, 2, 3; j = 1, \cdots, i$, are calculated from eq. (22) as

$$\begin{aligned} i_{11} &= \mathbf{p}_1^T \tilde{\mathbf{M}}_1 \mathbf{B}_{11} \mathbf{p}_1 = \mathbf{p}_1^T \tilde{\mathbf{M}}_1 \mathbf{p}_1 \\ &= I_1 + m_1 \mathbf{d}_1^T \mathbf{d}_1 + I_2 + m_2 \boldsymbol{\rho}_{12}^T \boldsymbol{\rho}_{12} + I_3 + m_3 \boldsymbol{\rho}_{13}^T \boldsymbol{\rho}_{13} = \frac{1}{3} m_1 l_1^2 + m_2 (l_1^2 + \\ &\quad \frac{1}{3} l_2^2 + l_1 l_2 C\theta_2) + m_3 (l_1^2 + l_2^2 + \frac{1}{3} l_3^2 + \\ &\quad 2 l_1 l_2 C\theta_2 + l_1 l_3 C\theta_{23} + l_2 l_3 C\theta_3) \end{aligned} \qquad (43a)$$

$$\begin{aligned} i_{21} &= i_{12} = \mathbf{p}_2^T \tilde{\mathbf{M}}_2 \mathbf{B}_{21} \mathbf{p}_1 \\ &= I_2 + m_2 \mathbf{d}_2^T \boldsymbol{\rho}_{12} + I_3 + m_3 \boldsymbol{\rho}_{23}^T \boldsymbol{\rho}_{13} \\ &= m_2 (\frac{1}{3} l_2^2 + \frac{1}{2} l_1 l_2 C\theta_2) + \\ &\quad m_3 (l_1 l_2 C\theta_2 + l_2^2 + l_2 l_3 C\theta_3 + \\ &\quad \frac{1}{2} l_1 l_3 C\theta_{23} + \frac{1}{3} l_3^2) \end{aligned} \qquad (43b)$$

$$\begin{aligned} i_{22} &= \mathbf{p}_2^T \tilde{\mathbf{M}}_2 \mathbf{B}_{22} \mathbf{p}_2 = \mathbf{p}_2^T \tilde{\mathbf{M}}_2 \mathbf{p}_2 \\ &= I_2 + m_2 \mathbf{d}_2^T \mathbf{d}_2 + I_3 + m_3 \boldsymbol{\rho}_{23}^T \boldsymbol{\rho}_{23} \\ &= \frac{1}{3} m_2 l_2^2 + m_3 (l_2^2 + \frac{1}{3} l_3^2 + l_2 l_3 C\theta_3) \end{aligned} \qquad (43c)$$

$$\begin{aligned} i_{31} &= i_{13} = \mathbf{p}_3^T \tilde{\mathbf{M}}_3 \mathbf{B}_{31} \mathbf{p}_1 = I_3 + m_3 \mathbf{d}_3^T \boldsymbol{\rho}_{13} \\ &= m_3 (\frac{1}{2} l_1 l_3 C\theta_{23} + \frac{1}{2} l_2 l_3 C\theta_3 + \frac{1}{3} l_3^2) \end{aligned} \qquad (43d)$$

$$\begin{aligned} i_{32} &= i_{23} = \mathbf{p}_3^T \tilde{\mathbf{M}}_3 \mathbf{B}_{32} \mathbf{p}_2 \\ &= I_3 + m_3 \mathbf{d}_3^T \boldsymbol{\rho}_{23} \\ &= m_3 (\frac{1}{2} l_2 l_3 C\theta_3 + \frac{1}{3} l_3^2) \end{aligned} \qquad (43e)$$

$$i_{33} = \mathbf{p}_3^T \tilde{\mathbf{M}}_3 \mathbf{B}_{33} \mathbf{p}_3 = \mathbf{p}_3^T \tilde{\mathbf{M}}_3 \mathbf{p}_3 = I_3 + m_3 \mathbf{d}_3^T \mathbf{d}_3 = \frac{1}{3} m_3 l_3^2, \quad (43f)$$

in which $\mathbf{B}_{11} = \mathbf{B}_{22} = \mathbf{B}_{33} = \mathbf{1}$, as pointed out in eq. (12), and the 6×6 matrices. $\mathbf{B}_{21}$ and $\mathbf{B}_{32}$, are as follows:

$$\mathbf{B}_{21} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{c}_{21} \times \mathbf{1} & \mathbf{1} \end{bmatrix} \text{ and } \quad \mathbf{B}_{32} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{c}_{32} \times \mathbf{1} & \mathbf{1} \end{bmatrix}$$

where $\mathbf{c}_{21} \times \mathbf{1}$ and $\mathbf{c}_{32} \times \mathbf{1}$ are the 3×3 cross-product tensors associated with the vectors, $\mathbf{c}_{21}$ and $\mathbf{c}_{32}$, respectively, as defined in eq. (11). The vectors $\mathbf{c}_{21}$ and $\mathbf{c}_{32}$ are given by

$$\mathbf{c}_{21} \equiv -\frac{1}{2}\left[l_2 \cos\theta_{12} + l_1 \cos\theta_1,\ l_2 \sin\theta_{12} + l_1 \sin\theta_1, 0\right]^T$$

$$\mathbf{c}_{32} \equiv -\frac{1}{2}\left[l_3 \cos\theta_{13} + l_2 \cos\theta_{12},\ l_3 \sin\theta_{13} + l_2 \sin\theta_{12}, 0\right]^T.$$

Moreover, the value of I_i—the moment of inertia of the ith link about the axis normal to the $X-Y$ plane (Fig. 3) and passing through C_i—is taken as, $I_i = \frac{1}{12} m_i l_i^2$, for $i = 1, 2, 3$. Furthermore,

$$\theta_{12} = \theta_1 + \theta_2; \quad \theta_{23} = \theta_2 + \theta_3; \quad \theta_{13} = \theta_1 + \theta_2 + \theta_3, \quad (44)$$

while $C(\cdot) \equiv \cos(\cdot)$; $S(\cdot) \equiv \sin(\cdot)$. In addition,

$$\tilde{\mathbf{M}}_1 = \mathbf{M}_1 + \mathbf{B}_{21}^T \tilde{\mathbf{M}}_2 \mathbf{B}_{21};$$
$$\tilde{\mathbf{M}}_2 = \mathbf{M}_2 + \mathbf{B}_{32}^T \tilde{\mathbf{M}}_3 \mathbf{B}_{32}; \quad \tilde{\mathbf{M}}_3 = \mathbf{M}_3$$

and

$$\mathbf{p}_1 \equiv \left[\mathbf{e}_1^T, (\mathbf{e}_1 \times \mathbf{d}_1)^T\right]^T \equiv \left[0, 0, 1, -\frac{1}{2} l_1 \sin\theta_1, \frac{1}{2} l_1 \cos\theta_1, 0\right]^T$$

$$\mathbf{p}_2 \equiv \left[\mathbf{e}_2^T, (\mathbf{e}_2 \times \mathbf{d}_2)^T\right]^T \equiv \left[0, 0, 1, -\frac{1}{2} l_2 \sin\theta_{12}, \frac{1}{2} l_2 \cos\theta_{12}, 0\right]^T$$

$$\mathbf{p}_3 \equiv \left[\mathbf{e}_3^T, (\mathbf{e}_3 \times \mathbf{d}_3)^T\right]^T \equiv \left[0, 0, 1, -\frac{1}{2} l_3 \sin\theta_{13}, \frac{1}{2} l_3 \cos\theta_{13}, 0\right]^T,$$

in which vectors $\mathbf{e}_i$ and $\mathbf{d}_i$ for $i = 1, 2, 3$ are

$$\mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_3 = \mathbf{k} = [0, 0, 1]^T,$$
$$\mathbf{d}_1 = \frac{1}{2} l_1 (C\theta_1 \mathbf{i} + S\theta_1 \mathbf{j}), \quad \mathbf{d}_2 = \frac{1}{2} l_2 (C\theta_{12} \mathbf{i} + S\theta_{12} \mathbf{j}),$$
$$\mathbf{d}_3 = \frac{1}{2} l_3 (C\theta_{13} \mathbf{i} + S\theta_{13} \mathbf{j}),$$

and the vectors ρ_{12}, ρ_{23}, and ρ_{13}, obtained from Figure 3, as

$$\rho_{12} = (l_1 C\theta_1 + \frac{1}{2} l_2 C\theta_{12})\mathbf{i} + (l_1 S\theta_1 + \frac{1}{2} l_2 S\theta_{12})\mathbf{j}$$
$$\rho_{23} = (l_2 C\theta_{23} + \frac{1}{2} l_3 C\theta_{13})\mathbf{i} + (l_2 S\theta_{12} + \frac{1}{2} l_3 S\theta_{13})\mathbf{j}$$
$$\rho_{13} = (l_1 C\theta_1 + l_2 C\theta_{12} + \frac{1}{2} l_3 C\theta_{13})\mathbf{i} + (l_1 S\theta_1 + l_2 S\theta_{12} + \frac{1}{2} l_3 S\theta_{13})\mathbf{j}.$$

The 3×3 decomposed matrices, $\mathbf{U}$ and $\mathbf{D}$, as derived in eq. (33), are now given as

$$\mathbf{U} \equiv \begin{bmatrix} 1 & \alpha_{12} & \alpha_{13} \\ & 1 & \alpha_{23} \\ 0's & & 1 \end{bmatrix} \text{ and } \quad (45)$$

$$\mathbf{D} \equiv \begin{bmatrix} \hat{m}_1 & & 0's \\ & \hat{m}_2 & \\ 0's & & \hat{m}_3 \end{bmatrix},$$

where $0's$ imply zeros, and

$$\alpha_{12} = \frac{1}{\hat{m}_2}[I_2 + m_2 \mathbf{d}_2^T \rho_{12} + I_3 + m_3 \rho_{13}^T \rho_{23} - \frac{1}{\hat{m}_3}(I_3 + m_3 \mathbf{d}_3^T \rho_{13})(I_3 + m_3 \mathbf{d}_3^T \rho_{23})],$$

$$\alpha_{13} = \frac{1}{\hat{m}_3}(I_3 + m_3 \mathbf{d}_3^T \rho_{13}), \quad \alpha_{23} = \frac{1}{\hat{m}_3}(I_3 + m_3 \mathbf{d}_3^T \rho_{23}),$$

and

$$\hat{m}_1 = I_1 + m_1 \mathbf{d}_1^T \mathbf{d}_1 + I_2 + m_2 \rho_{12}^T \rho_{12} + I_3 + m_3 \rho_{13}^T \rho_{13} - \frac{1}{\hat{m}_3}(I_3 + m_3 \mathbf{d}_3^T \rho_{13})^2 - \frac{1}{\hat{m}_2}[I_2 + m_2 \mathbf{d}_2^T \rho_{12} + I_3 + m_3 \rho_{23}^T \rho_{13} - \frac{1}{\hat{m}_3}(I_3 + m_3 \mathbf{d}_3^T \rho_{13})(I_3 + m_3 \mathbf{d}_3^T \rho_{23})]^2$$

$$\hat{m}_2 = I_2 + m_2 \mathbf{d}_2^T \mathbf{d}_2 + I_3 + m_3 \rho_{23}^T \rho_{23} - \frac{1}{\hat{m}_3}(I_3 + m_3 \mathbf{d}_3^T \rho_{23})^2$$

$$\hat{m}_3 = i_{33} = I_3 + m_3 \mathbf{d}_3^T \mathbf{d}_3.$$

The expressions for the elements of the 3 × 3 inertia matrix, I, namely, eqs. (43a)–(43f), can also be verified from the $\mathbf{UDU}^T$ decomposition of I, eq. (45), that is,

$$\mathbf{I} \equiv \begin{bmatrix} \hat{m}_1 + \hat{m}_2\alpha_{12}^2 + \hat{m}_3\alpha_{13}^2 & & \text{sym} \\ \hat{m}_2\alpha_{12} + \hat{m}_3\alpha_{13}\alpha_{23} & \hat{m}_2 + \hat{m}_3\alpha_{23}^2 & \\ \hat{m}_3\alpha_{13} & \hat{m}_3\alpha_{23} & \hat{m}_3 \end{bmatrix}. \quad (46)$$

6.2. *The* 3 × 3 *Inverse*, $\mathbf{I}^{-1}$

The inverse of the 3 × 3 matrices, U and D, namely, $\mathbf{U}^{-1}$ and $\mathbf{D}^{-1}$, are

$$\mathbf{U} \equiv \begin{bmatrix} 1 & \alpha'_{12} & \alpha'_{13} \\ & 1 & \alpha'_{23} \\ 0's & & 1 \end{bmatrix} \quad \text{and} \quad (47)$$

$$\mathbf{D} \equiv \begin{bmatrix} \frac{1}{\hat{m}_1} & & 0's \\ & \frac{1}{\hat{m}_2} & \\ 0's & & \frac{1}{\hat{m}_3} \end{bmatrix},$$

where

$$\alpha'_{12} = \alpha_{12}, \quad \alpha'_{23} = \alpha_{23},$$

$$\alpha'_{13} = \frac{1}{\hat{m}_3}\Big[I_3 + m_3\mathbf{d}_3^T\rho_{13} - \frac{1}{\hat{m}_2}\{I_2 + m_2\mathbf{d}_2^T\rho_{12} + I_3 + m_3\rho_{23}^T\rho_{13} - \frac{1}{\hat{m}_3}(I_3 + m_3\mathbf{d}_3^T\rho_{13}) (I_3 + m_3\mathbf{d}_3^T\rho_{23})\}\ (I_3 + m_3\mathbf{d}_3^T\rho_{23})\Big].$$

Finally, the expression for elements of the 3 × 3 matrix, $\mathbf{I}^{-1}$, is as follows:

$$\mathbf{I}^{-1} \equiv \begin{bmatrix} \frac{1}{\hat{m}_1} & & \text{sym} \\ -\frac{1}{\hat{m}_1}\alpha'_{12} & \frac{1}{\hat{m}_1}\alpha'^2_{12} + \frac{1}{\hat{m}_2} & \\ -\frac{1}{\hat{m}_1}\alpha'_{13} & \frac{1}{\hat{m}_1}\alpha'_{12}\alpha'_{13} - \frac{1}{\hat{m}_2}\alpha'_{23} & \frac{1}{\hat{m}_1}\alpha'^2_{13} + \frac{1}{\hat{m}_2}\alpha'^2_{23} + \frac{1}{\hat{m}_3} \end{bmatrix}. \quad (48)$$

It is now a simple matter to use eqs. (46) and (48) to verify that $\mathbf{II}^{-1} = \mathbf{1}$.

7. Conclusions

Based on the Gaussian elimination of the inertia matrix of a serial robot, analytical expressions for the inverse of the inertia matrix are obtained. This is possible due to the derivation of the dynamic equations of motion using the Decoupled Natural Orthogonal Complement matrices. The approach led to a recursive $\mathcal{O}(n)$ dynamics algorithm (Saha 1997), which is omitted from this paper because the emphasis here is on how to get the inverted inertia matrix. The complexity of the algorithm, as reported in Saha (1997), is $(201n\text{-}335)M(193n\text{-}361)A$ (M and A imply "multiplication/division" and "addition/subtraction," respectively) compared to $(199n - 198)M$ and $(174n - 173)A$, required by the algorithm of Featherstone (1983). No complexity is, however, available for the approach by Rodriguez and Kreutz-Delgado (1992).

Appendix

In the *reverse* Gaussian elimination, after annihilation of the first $(n-1)$ elements of the nth column, the modified inertia matrix, denoted by $\mathbf{I}^{(n)}$, is given by

$$\mathbf{I}^{(n)} \equiv \begin{bmatrix} i_{11}^{(n)} & & \text{sym} & 0 \\ \vdots & \ddots & & 0 \\ i_{n-1,1}^{(n)} & \cdots & i_{n-1,n-1}^{(n)} & 0 \\ i_{n1} & \cdots & i_{n,n-1} & i_{nn} \end{bmatrix}, \quad (49)$$

where i_{nn} is the *pivot* (Stewart 1973), and the scalars, $i_{ij}^{(n)}$, are the modified elements of I, whereas "sym" denotes the symmetric elements of the $(n-1) \times (n-1)$ matrix, resulting from the deletion of the nth row and column of the matrix, $\mathbf{I}^{(n)}$. The elements $i_{ij}^{(n)}$ are calculated from the following formula while $k = n$:

$$i_{ij}^{(k)} = \mathbf{p}_i^T \hat{\mathbf{M}}_i^{(k)} \mathbf{B}_{ij}\, \mathbf{p}_j \quad (50)$$

where the 6 × 6 symmetric matrix $\hat{\mathbf{M}}_i^{(k)}$ is given by

$$\hat{\mathbf{M}}_i^{(k)} = \mathbf{M}_i + \mathbf{B}_{i+1,i}^T \hat{\mathbf{M}}_{i+1}^{(k)} \mathbf{B}_{i+1,i}$$

in which $\hat{\mathbf{M}}_i^{(i)} \equiv \mathbf{\Psi}_i\hat{\mathbf{M}}_i$ and $\hat{\mathbf{M}}_i^{(i+1)} \equiv \hat{\mathbf{M}}_i$ that are defined in eq. (35).

Equation (49) can also be realized by premultiplying the matrix I with the *elementary upper triangular matrix* (EUTM), $\mathbf{E}_k$, as defined in Section 5.1, for $k = n$; that is, $\mathbf{I}^{(n)} = \mathbf{E}_n\mathbf{I}$, where $\mathbf{E}_n$ has the following structure:

$$\mathbf{E}_n \equiv \begin{bmatrix} 1 & \cdots & 0 & -\alpha_{1n} \\ & \ddots & \vdots & \vdots \\ & & 1 & -\alpha_{n-1,n} \\ & & & 1 \end{bmatrix}. \quad (51)$$

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