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Dynamic Model Simplification of Serial Manipulators

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***Abstract* – Simplicity in the dynamics model of a serial robot manipulator greatly enhances the speed of its control and the associated hardware implementation. Since the motion of one link influences the torque or force required at the other joints, the control becomes difficult. This is referred as dynamic coupling. In this paper, it is proposed to simplify the robot's dynamic coupling by suitably choosing the manipulator's kinematic and dynamic parameters. The intention is to make the Generalized Inertia Matrix (GIM) of the serial manipulator associated with its dynamic equations of motion diagonal and/or constant. Such choice automatically ensures the associated convective inertia terms vanish. Such simplifications are carried out by investigating the expression of each element of the GIM. The concepts of the twist propagation matrices and the joint-motion propagation vectors are used to obtain the analytical expressions of the GIM elements that allow one to investigate the elements for simplifications. The methodology is illustrated with a 3-link spatial manipulator arm.**

***Index Terms* – Manipulator, Dynamics, Simplification.**

I. INTRODUCTION

Dynamic equations of motion of a serial-type robotic manipulator are generally complex where the motion of one link affects the others, i.e., there are coupled terms in the equations of motion. Such complexity not only makes the control algorithm difficult but also slows down the speed of the robot due to the computational complexity of the robot dynamics [1]. There have been long efforts in improving the dynamic modeling methodologies, e.g., [1-5] and others, in order to reduce the on-line computational time of the dynamics and, as a result, enhance the robot speed. Alternatively, [6-9] have looked into the problem differently, where they wanted to design a robot with simpler dynamics. Some of these lead to the design based on mass balancing, e.g., in [10,11]. Whereas the mass balancing is generally carried out to reduce the shaking forces and moments transmitted to the fixed-base in order to reduce the ground vibrations, the simplification of the dynamics is intended to simplify the control. The papers in [1-5] and the present paper focus on the latter aspect. In this paper, dynamic equations of motion of a serial robot manipulator are derived using the twist propagation matrices and joint-motion propagation vectors. The use of the concept of the twist and joint-motion propagations allows one to write the elements of the associated matrices and vectors in analytical form. Then, the elements of the Generalized Inertia Matrix (GIM) are made zero or constant through the appropriate choice of the manipulator's kinematic and dynamic parameters like the location of the mass center, etc. so that the GIM becomes constant diagonal or constant. The results obtained in this paper are in agreement with those reported in [7] and others, however, the proposed methodology differ in the following manners: 1) instead of using complex geometrical concepts such as immobilizing the joints and then defining a rigid body, etc., a straight-forward simple velocity transformations, namely, the *twist* and *joint-motion propagations*, are used; 2) using the concepts of the above propagations, analytical expressions for the elements of the matrices associated with the Newton-Euler dynamic equations of motion are obtained, which are then inspected to simplify the dynamics; 3) the use of the propagation matrices and vectors that are the basic constituents of the Decoupled Natural Orthogonal Complement (DeNOC) matrices play the key roles in obtaining both the recursive inverse and forward dynamics algorithms for serial and parallel manipulators in a unified way, as reported in [5] and others.

Even though the research in the area of manipulator parameters for simpler dynamics has initiated during 1980s not much publications have been reported in recent times. This could be due to the availability of high-speed computer processors to achieve a real-time control of a manipulator, which was almost impractical in earlier days. Note, however, that besides the real-time control, the dynamic simplification also improves the numerical stability. As a result, wherever simplification of a robot dynamics is possible it should be pursued, which has been the motivation for this research work.

This paper is organized as follows: Section II presents how to obtain the analytical expressions for the elements of the Generalized Inertia Matrix (GIM) associated with the dynamic equations of motion, which will be investigated to make them constant or vanish. Section III illustrates the simplification of the dynamics, namely, the GIM, by rearranging the parameters of a 3-link spatial manipulator arm. Finally, conclusions are given in Section IV.

II. ANALYTICAL EXPRESSIONS

For an n-degree of freedom open-loop serial-chain robot, as shown in Fig. 1, the dynamic equations of motion can be represented as:

$$\mathbf{I}\,\ddot{\boldsymbol{\theta}} + \mathbf{C}\,\dot{\boldsymbol{\theta}} = \boldsymbol{\tau} \tag{1}$$

where the $n \times n$ Generalized Inertia Matrix (GIM), $\mathbf{I}$, the $n \times n$ matrix of the convective inertia terms, $\mathbf{C}$, and the n-dimensional vector of the generalized forces due to external driving torques, gravity, etc., $\boldsymbol{\tau}$, are written from the uncoupled Newton-Euler (NE) equations of motion and the Decoupled Natural Orthogonal Complement (DeNOC) matrices [5] as

$$\mathbf{I} \equiv \mathbf{T}_d^T \tilde{\mathbf{M}} \mathbf{T}_d \,;\; \mathbf{C} \equiv \mathbf{T}_d^T (\mathbf{T}_l^T \mathbf{M} \dot{\mathbf{T}}_l + \tilde{\mathbf{M}} \mathbf{W} + \dot{\tilde{\mathbf{M}}}) \mathbf{T}_d \,;\text{ and}$$
$$\boldsymbol{\tau} \equiv \mathbf{T}_d^T \tilde{\mathbf{w}}^E \tag{2a}$$

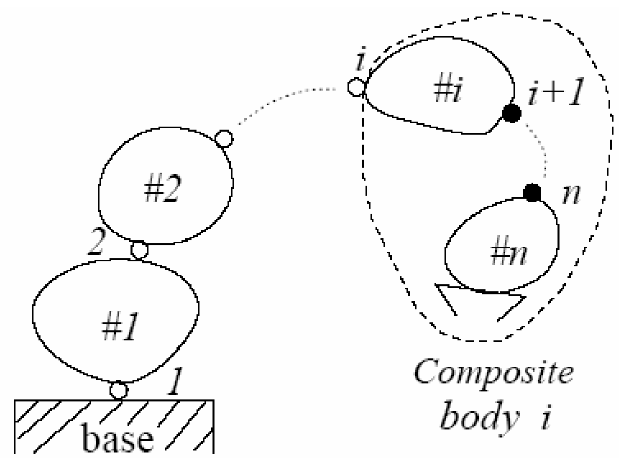

Fig. 1 An n-link serial-type robot manipulator

The $6n \times 6n$ matrices, $\tilde{\mathbf{M}}$, $\dot{\tilde{\mathbf{M}}}$, and the $6n$-dimensional vector, $\tilde{\mathbf{w}}^{E}$ of eq. (2a), are given by

$$\tilde{\mathbf{M}} \equiv \mathbf{T}_l^T \mathbf{M} \mathbf{T}_l \,;\; \dot{\tilde{\mathbf{M}}} \equiv \mathbf{T}_l^T \mathbf{W} \mathbf{M} \mathbf{E} \mathbf{T}_l \,;\text{ and } \tilde{\mathbf{w}}^{E} \equiv \mathbf{T}_l^T \mathbf{w}^{E} \tag{2b}$$

where the $6n \times 6n$ lower block triangular matrix, $\mathbf{T}_l$, and the $6n \times n$ block diagonal matrix, $\mathbf{T}_d$, are defined as

$$\mathbf{T}_l \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{A}_{21} & \mathbf{1} & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{n1} & \mathbf{A}_{n2} & \cdots & \mathbf{1} \end{bmatrix} ;\; \mathbf{T}_d \equiv \begin{bmatrix} \mathbf{p}_1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{p}_2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{p}_n \end{bmatrix} \tag{3}$$

in which the 6×6 *twist-propagation* matrix, $\mathbf{A}_{ij}$, and the 6-dimensional *joint-motion propagation* vector, $\mathbf{p}_i$, are obtained from the velocity relations of the coupled bodies, as shown in Fig. 2, i.e.,

$$\mathbf{t}_i = \mathbf{A}_{i,i-1}\,\mathbf{t}_{i-1} + \mathbf{p}_i\,\dot{\theta}_i \tag{4}$$

Matrix $\mathbf{A}_{i,i-1}$ and vector $\mathbf{p}_i$ are given by

$$\mathbf{A}_{i,i-1} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ -\mathbf{a}_{i,i-1} \times \mathbf{1} & \mathbf{1} \end{bmatrix} ;\text{ and} \tag{5a}$$

$$\mathbf{p}_i \equiv \begin{bmatrix} \mathbf{e}_i \\ \mathbf{0} \end{bmatrix} : \text{For revolute;} \quad \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_i \end{bmatrix} : \text{For prismatic} \tag{5b}$$

Moreover, vector $\mathbf{e}_i$ is the unit vector parallel to the axis of rotation of the i^{th} revolute joint, or the direction of translation of the i^{th} prismatic joint. In eq. (5a), $-\mathbf{a}_{i,i-1} \times \mathbf{1}$ is the 3×3 cross-product tensor associated with the vector, $-\mathbf{a}_{i,i-1} = \mathbf{a}_{i-1,i}$. Vector $\mathbf{a}_{i-1,i}$ is shown in Fig. 2, which when operates on the 3-dimensional Cartesian vector, $\mathbf{x}$, results in the cross-product vector, $(\mathbf{a}_{i-1,i} \times \mathbf{1})\mathbf{x} = \mathbf{a}_{i-1,i} \times \mathbf{x}$. Note that, for three successively coupled rigid bodies, say, #(i -1), #i, and #(i + 1), Fig. 2, the twist propagation matrices obey the following properties:

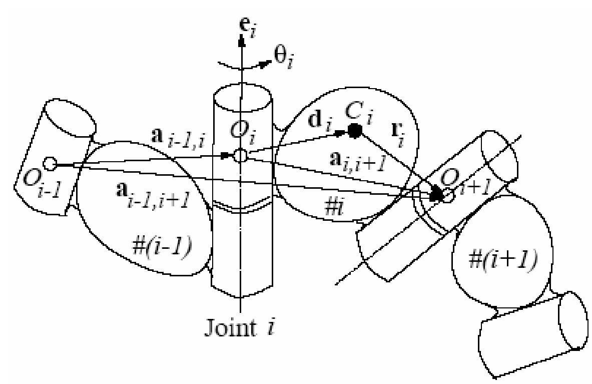

Fig. 2 Three serially coupled bodies

$$\mathbf{A}_{i-1,i}\,\mathbf{A}_{i,i+1} = \mathbf{A}_{i-1,i+1};\; \mathbf{A}_{ii} = \mathbf{1};\text{ and}$$
$$\mathbf{A}^{-1}_{i-1,i} = \mathbf{A}_{i,i-1} \text{ or } \mathbf{A}^{-1}_{i,i+1} = \mathbf{A}_{i+1,i} \tag{6}$$

where $\mathbf{A}_{i-1,i+1}$ is the function of the vector, $\mathbf{a}_{i-1,i+1}$, denoting the distance from the origin of the $(i\,-1)^{\text{st}}$ link, O_{i-1}, to the origin of the $(i+1)^{\text{st}}$ one, O_{i+1}. Hence, $\mathbf{A}_{ii}$ is the identity matrix, as the distance from the origin of the i^{th} link to itself vanishes. Using the properties given in eq. (6), and the expressions of eqs. (2a&b), each element of the GIM, **I**, is given by

$$i_{ij} \equiv \mathbf{p}_i^{\mathrm{T}} \tilde{\mathbf{M}}_i \mathbf{A}_{ij} \mathbf{p}_j \qquad (7)$$

where the 6 × 6 matrix, $\tilde{\mathbf{M}}_i$, can be written as

$$\tilde{\mathbf{M}}_i \equiv \mathbf{M}_i + \mathbf{A}_{i+1,i}^{\mathrm{T}} \tilde{\mathbf{M}}_{i+1} \mathbf{A}_{i+1,i} \qquad (8)$$

for $i = n, \ldots, 1$. Note that, for $i = n$, $\tilde{\mathbf{M}}_n \equiv \mathbf{M}_n$, as there is no $(n+1)^{\mathrm{st}}$ link in the serial chain, i.e., $\tilde{\mathbf{M}}_{n+1} \equiv \mathbf{O}$. Moreover,

$$\tilde{\mathbf{M}}_{n-1} \equiv \mathbf{M}_{n-1} + \mathbf{A}_{n,n-1}^{\mathrm{T}} \mathbf{M}_n \mathbf{A}_{n,n-1} \qquad (9)$$

which is interpreted as the mass matrix of the "composite body, (n-1)" formed by rigidly joining the bodies, #n and #(n-1). For the composite body, i, it is indicated in Fig. 1. Matrix $\tilde{\mathbf{M}}_i$ in eq. (8) is now expressed in terms of its 3 × 3 block matrices, namely,

$$\tilde{\mathbf{M}}_i = \begin{bmatrix} \tilde{\mathbf{I}}_i & sym \\ -\tilde{\boldsymbol{\delta}}_i \times 1 & \tilde{m}_i \mathbf{1} \end{bmatrix} \qquad (10)$$

where "*sym*" denotes the symmetric elements of the composite mass matrix, $\tilde{\mathbf{M}}_i$, and the 3×3 inertia tensor of the i^{th} composite body, $\tilde{\mathbf{I}}_i$, its mass, $\tilde{m}_i$, and the 3-dimensional vector, $\tilde{\boldsymbol{\delta}}_i$, are given by

$$\tilde{\mathbf{I}}_i \equiv \sum_{k=i}^{n} \left[\mathbf{I}_k^{\mathrm{C}} - m_k \boldsymbol{\rho}_{ik} \times (\boldsymbol{\rho}_{ik} \times \mathbf{1}) \right] \qquad (11a)$$

$$\tilde{m}_i \equiv \sum_{k=i}^{n} m_k \; ; \; \tilde{\boldsymbol{\delta}}_i \equiv \sum_{k=i}^{n} m_j \boldsymbol{\rho}_{ij} \qquad (11b)$$

Also, the vector, $\boldsymbol{\rho}_{ik} \equiv \mathbf{a}_i + \mathbf{a}_{i+1} + \cdots + \mathbf{a}_{k-1} + \mathbf{d}_k$, for $i<k$; and $\boldsymbol{\rho}_{ii} \equiv \mathbf{d}_i$, for $i = k$, in eqs. (11a&b), which represents the position of the mass center of the k^{th} body, C_k, from the origin of the i^{th} one, i.e., O_i., as indicated in Fig. 3 for $\boldsymbol{\rho}_{13}$. Matrix, $\tilde{\mathbf{I}}_i$, eq. (11a), is interpreted as the inertia tensor of the "*composite body, i*," Fig. 1, about the origin of the i^{th} body, O_i, which is nothing but the inertia matrix of the *rigid-body* of [7] while the last (n-i+1) joints are immobilized. In [7], the matrix was obtained from the Euler-Lagrange equations through complex partial differentiations, whereas in this paper it is obtained from the uncoupled NE equations through a set of simple matrix operations only. Equations (11a&b) are the key expressions for the simplification of the robot dynamics. Moreover, eq. (10) plays an important role in obtaining the recursive inverse and forward dynamics algorithms [5], as pointed out in the Introduction.

III. DYNAMIC SIMPLIFICATION

The objective here is to choose the manipulator parameters in such a way that the GIM is invariant, i.e., the elements of the GIM given by eq. (7) are constant. Even better would be to make the GIM diagonal for which the control for the robot is decoupled. This not only enhances the speed of the robot but simplifying the control design to achieve stability. It can be shown that if the elements of the GIM are constant, the matrix of the convective inertia terms, **C** of eq. (2), vanishes. Hence, the dynamic simplification problem reduces to the investigation of the explicit expressions of the GIM, and find out the conditions so that they are constant for the diagonal elements, and constant or zero for the off-diagonal elements. Note here that if two revolute joint axes are parallel the corresponding element of the GIM is never zero that can be shown as follows: Using eqs. (2a&b), (5) and (10), the (i,j)th element can be expanded as

$$i_{ij} \equiv \mathbf{e}_i^{\mathrm{T}} \tilde{\mathbf{I}}_i \mathbf{e}_j = \sum_{k=i}^{n} [\mathbf{e}_i^{\mathrm{T}} \mathbf{I}_k^C \mathbf{e}_j + m_k (\mathbf{e}_i \times \mathbf{d}_k)^{\mathrm{T}} (\mathbf{e}_j \times \boldsymbol{\rho}_{ik})] \qquad (12)$$

From numerical computation point of view, while $\mathbf{e}_i$ and $\mathbf{e}_j$ are parallel, it will have the same non-zero elements depending on a coordinate frame chosen to represent all the vectors of eq. (12). If a coordinate frame is chosen where $\mathbf{e}_i = \mathbf{e}_i \equiv [0,0,1]^{\mathrm{T}}$, then i_{ij} is nothing but the (3,3) element of the matrix $\tilde{\mathbf{I}}_i$. Referring to eq. (12), even if the manipulator parameters are such that the second term of the right hand expression containing vectors $\mathbf{d}_k$ and ρ_{ik} vanish, e.g., when $\mathbf{d}_k$ and ρ_{ik} are always orthogonal, but the (3,3) element of $\mathbf{I}_k^C$ will not vanish unless the mass of the associated composite links lie at the origin of joint k. This will lead to an impractical manipulator arm, which is also obvious from the expressions of eqs. (14&b), as derived later. The same interpretation was obtained in [7] after several propositions and theorems were introduced, whereas it is a simple observation from eq. (12). Hence, the power of the present formulation using the twist and joint-motion propagations is established.

Next, the dynamic simplification with respect to a 3-link spatial manipulator arm with all revolute joints, as shown in Fig. 3, will be shown. Each element of the 3×3 GIM, **I**, is given by eq. (12), for i = 3, 2, 1; j = i, …, 1. Using the definition of the dot-product of two cross-product vectors, the six lower triangular elements of the symmetric GIM, i_{ij}, for the spatial 3-link arm, as shown in Fig. 3, are written as

$$i_{33} = \mathbf{e}_3^T \left[\mathbf{I}_3^C + m_3 (\mathbf{d}_3^{\mathrm{T}} \mathbf{d}_3 \mathbf{1} - \mathbf{d}_3 \mathbf{d}_3^{\mathrm{T}}) \right] \mathbf{e}_3 \qquad (13a)$$

$$i_{32} = \mathbf{e}_3^T\left[\mathbf{I}_3^C + m_3(\boldsymbol{\rho}_{23}^T\mathbf{d}_3\mathbf{1} - \boldsymbol{\rho}_{23}\mathbf{d}_3^T)\right]\mathbf{e}_2 \quad (13b)$$

$$i_{31} = \mathbf{e}_3^T\left[\mathbf{I}_3^C + m_3(\boldsymbol{\rho}_{13}^T\mathbf{d}_3\mathbf{1} - \boldsymbol{\rho}_{13}\mathbf{d}_3^T)\right]\mathbf{e}_1 \quad (13c)$$

$$i_{22} = \mathbf{e}_2^T\begin{bmatrix}\mathbf{I}_2^C + m_2(\mathbf{d}_2^T\mathbf{d}_2\mathbf{1} - \mathbf{d}_2\mathbf{d}_2^T) + \\ \mathbf{I}_3^C + m_3(\boldsymbol{\rho}_{23}^T\boldsymbol{\rho}_{23}\mathbf{1} - \boldsymbol{\rho}_{23}\boldsymbol{\rho}_{23}^T)\end{bmatrix}\mathbf{e}_2 \quad (13d)$$

$$i_{21} = \mathbf{e}_2^T\begin{bmatrix}\mathbf{I}_2^C + m_2(\boldsymbol{\rho}_{12}^T\mathbf{d}_2\mathbf{1} - \boldsymbol{\rho}_{12}\mathbf{d}_2^T) + \\ \mathbf{I}_3^C + m_3(\boldsymbol{\rho}_{12}^T\boldsymbol{\rho}_{23}\mathbf{1} - \boldsymbol{\rho}_{12}\boldsymbol{\rho}_{23}^T)\end{bmatrix}\mathbf{e}_1 \quad (13e)$$

$$i_{11} = \mathbf{e}_1^T\begin{bmatrix}\mathbf{I}_1^C + m_1(\mathbf{d}_1^T\mathbf{d}_1\mathbf{1} - \mathbf{d}_1\mathbf{d}_1^T) + \\ \mathbf{I}_2^C + m_2(\boldsymbol{\rho}_{12}^T\boldsymbol{\rho}_{12}\mathbf{1} - \boldsymbol{\rho}_{12}\boldsymbol{\rho}_{12}^T) + \\ \mathbf{I}_3^C + m_3(\boldsymbol{\rho}_{13}^T\boldsymbol{\rho}_{13}\mathbf{1} - \boldsymbol{\rho}_{13}\boldsymbol{\rho}_{13}^T)\end{bmatrix}\mathbf{e}_1 \quad (13f)$$

The primary objective here is to make the diagonal elements, namely, i_{33}, i_{22}, and i_{11}, constant, and the off-diagonal elements, i.e., i_{32}, i_{31}, and i_{21}, either zero or constant so that the robot arm is configuration-independent. As a result the control becomes simple. In case of zero off-diagonal elements, the robot control becomes equivalent to the independent control of each joint actuator, i.e., there is no dynamic coupling, an inherent difficulty in the control of serial robots. Each element will be checked for its vanishing or constant condition, and the corresponding constraints will form the basis for the modified design parameters for simplified dynamics. This is done through investigation of each term one by one, as explained next.

Step 1: Check for i_{33}

Representing the matrix, $\mathbf{I}_3^C$, and the vector, $\mathbf{d}_3$, in its own frame, i.e., Frame 3, as represented in Fig. 3, one gets a constant number, namely,

$$i_{33} = \mathbf{e}_3^T\mathbf{I}_3^C\mathbf{e}_3 + m_3[(\mathbf{d}_3^T\mathbf{d}_3)(\mathbf{e}_3^T\mathbf{e}_3) - (\mathbf{e}_3^T\mathbf{d}_3)^2]$$
$$= I_3^{ZZ} + m_3 d_3^2 \quad = \text{constant} \quad (14a)$$

where $I_3^{ZZ} = m_3 a_3^2/12$ ---a_3 being the length of link 3---is the principle mass moment of inertia of the link 3 about its *Z*-axis passing through the mass center, C_3, From Fig. 3, $\mathbf{e}_3^T\mathbf{d}_3 = 0$, as $\mathbf{e}_3$ is orthogonal to $\mathbf{d}_3$; $\mathbf{e}_3^T\mathbf{e}_3 = 1$, as $\mathbf{e}_3$ is a unit vector; $\mathbf{d}_3^T\mathbf{d}_3 = d_3^2$; and $\mathbf{e}_3^T\mathbf{I}_3^C\mathbf{e}_3 = I_3^{ZZ}$, as $\mathbf{e}_3$ in Frame 3 is, $[\mathbf{e}_3]_3=[0,0,1]^T$. Moreover, d_3 is the distance of the mass center from the origin point, O_3, as indicated in Fig. 3. The term, i_{33}, in eq. (14a) is interpreted as the moment of inertia of link 3 about the axis of joint 3, i.e., Z_3, passing through O_3.

Step 2: Check for i_{32}

The term, i_{32} of eq. (13b), can be written as

$$i_{32} = \mathbf{e}_3^T\mathbf{I}_3^C\mathbf{e}_2 + m_3[(\boldsymbol{\rho}_{23}^T\mathbf{d}_3)(\mathbf{e}_3^T\mathbf{e}_2) - (\mathbf{e}_3^T\boldsymbol{\rho}_{23})(\mathbf{e}_2^T\mathbf{d}_3)]$$
$$= I_3^{ZZ} + m_3(d_2^2 + a_2 d_3 C_3) \quad (14b)$$

where the unit vectors, $\mathbf{e}_2$ and $\mathbf{e}_3$, denoting the axes of rotation of the 2nd and 3rd revolute joints, respectively, are parallel. Hence, $\mathbf{e}_3^T\mathbf{I}_3^C\mathbf{e}_2 = I_3^{ZZ}$. Moreover, $C_3 \equiv Cos\theta_3$---θ_3 being the angle of rotation of joint 3. In eq. (14b), the only variable quantity is, $a_2d_3C_3$, due to the term, C_3, which varies as the robot moves. As a result, either a_2 or d_3 has to be zero. Since $a_2 = 0$ implies a smaller workspace of the robot, d_3=0 is preferred which can be achieved by changing the shape of the 3rd link, as indicated in Fig. 4.

Step 3: Check for i_{31}

The term, i_{31} of eq. (13c), is re-written as

$$i_{31} = \mathbf{e}_3^T\mathbf{I}_3^C\mathbf{e}_1 + m_3[(\boldsymbol{\rho}_{13}^T\mathbf{d}_3)(\mathbf{e}_3^T\mathbf{e}_1) - (\mathbf{e}_3^T\boldsymbol{\rho}_{13})(\mathbf{e}_1^T\mathbf{d}_3)]$$
$$= 0 \quad (14c)$$

where the unit vectors, $\mathbf{e}_1$ and $\mathbf{e}_3$, i.e., those parallel to the axes of joints 1 and 3, respectively, are orthogonal to each other. Hence, $\mathbf{e}_3^T\mathbf{e}_1 = 0$. Moreover, the vector, $\mathbf{e}_3$, is always orthogonal to vector $\boldsymbol{\rho}_{13}$, as clear from Fig. 3. This yields, $\mathbf{e}_3^T\boldsymbol{\rho}_{13} = 0$. Furthermore, the inertia tensor, $\mathbf{I}_3^C$, can be chosen so that it is diagonal in its own frame, i.e., Frame 3. As a result, the vector term, $\mathbf{I}_3^C\mathbf{e}_3$, is parallel to $\mathbf{e}_3$, and $\mathbf{e}_3^T\mathbf{I}_3^C\mathbf{e}_1 = (\mathbf{I}_3^C\mathbf{e}_3)^T\mathbf{e}_1 = 0$.

Step 4: Check for i_{22}

Representing the matrix, $\mathbf{I}_3^C$, and the vector, $\mathbf{d}_3$, in its own frame, i.e., Frame 3, one gets a constant number, namely,

$$i_{22} = \mathbf{e}_2^T\mathbf{I}_2^C\mathbf{e}_2 + m_2[(\mathbf{d}_2^T\mathbf{d}_2)(\mathbf{e}_2^T\mathbf{e}_2) - (\mathbf{e}_2^T\mathbf{d}_2)^2] +$$
$$\mathbf{e}_2^T\mathbf{I}_3^C\mathbf{e}_2 + m_3[(\boldsymbol{\rho}_{23}^T\boldsymbol{\rho}_{23})(\mathbf{e}_2^T\mathbf{e}_2) - (\mathbf{e}_2^T\boldsymbol{\rho}_{23})^2]$$
$$= I_2^{ZZ} + m_3 d_3^2 + I_3^{ZZ} + m_3(a_2^2 + d_3^2 + 2a_2 d_3 C_3)$$
$$= \text{constant} \quad (14d)$$

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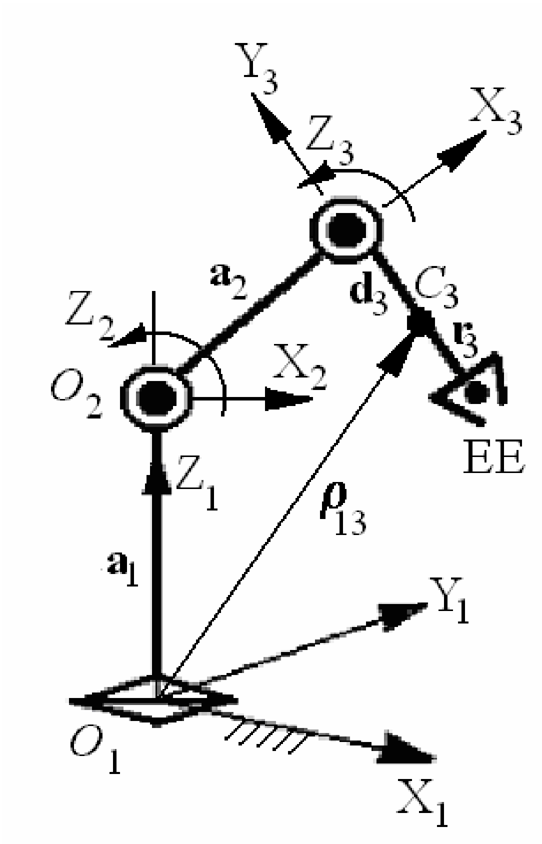

Fig. 3 A 3-link all revolute spatial manipulator arm

as $\mathbf{e}_2^T\boldsymbol{\rho}_{23} = \mathbf{e}_2^T\mathbf{d}_2 = 0$, and the only variable quantity, $a_2d_3C_3$, vanishes because $d_3 = 0$, which is the condition obtained in Step 2.

Step 5: Check for i_{21}

The term, i_{21} of eq. (13e), is re-written as

$$
\begin{aligned}
i_{21} &= \mathbf{e}_2^T\mathbf{I}_2^C\mathbf{e}_1 + m_2[(\boldsymbol{\rho}_{12}^{\mathrm{T}}\mathbf{d}_2)(\mathbf{e}_2^T\mathbf{e}_1) - (\mathbf{e}_2^T\boldsymbol{\rho}_{12})(\mathbf{e}_1^T\mathbf{d}_2)] + \\
&\quad \mathbf{e}_2^T\mathbf{I}_3^C\mathbf{e}_1 + m_3[(\boldsymbol{\rho}_{13}^{\mathrm{T}}\boldsymbol{\rho}_{23})(\mathbf{e}_2^T\mathbf{e}_1) - (\mathbf{e}_2^T\boldsymbol{\rho}_{13})(\mathbf{e}_1^T\boldsymbol{\rho}_{23})] \\
&= 0
\end{aligned}
\tag{14e}
$$

since $\mathbf{e}_2^T\mathbf{e}_1 = \mathbf{e}_2^T\boldsymbol{\rho}_{23} = \mathbf{e}_2^T\boldsymbol{\rho}_{13} = 0$. Moreover, the matrices, $\mathbf{I}_2^C$ and $\mathbf{I}_3^C$ are diagonal. Hence, the vector terms, $\mathbf{I}_2^C\mathbf{e}_2$ and $\mathbf{I}_3^C\mathbf{e}_2$, are parallel to each other, and orthogonal to vector $\mathbf{e}_1$. This implies, $\mathbf{e}_2^T\mathbf{I}_2^C\mathbf{e}_1 = (\mathbf{I}_2^C\mathbf{e}_2)^T\mathbf{e}_1 = 0$, and $\mathbf{e}_2^T\mathbf{I}_3^C\mathbf{e}_1 = (\mathbf{I}_3^C\mathbf{e}_2)^T\mathbf{e}_1 = 0$.

Step 6: Check for i_{11}

Expanding the expressions for i_{11} from eq. (13f), one obtains the following:

$$
\begin{aligned}
i_{11} &= \mathbf{e}_1^T\mathbf{I}_1^C\mathbf{e}_1 + m_1[(\mathbf{d}_1^{\mathrm{T}}\mathbf{d}_1)(\mathbf{e}_1^T\mathbf{e}_1) - (\mathbf{e}_1^T\mathbf{d}_1)^2] + \\
&\quad \mathbf{e}_1^T\mathbf{I}_2^C\mathbf{e}_1 + m_2[(\boldsymbol{\rho}_{12}^{\mathrm{T}}\boldsymbol{\rho}_{12})(\mathbf{e}_1^T\mathbf{e}_1) - (\mathbf{e}_1^T\boldsymbol{\rho}_{12})^2] + \\
&\quad \mathbf{e}_1^T\mathbf{I}_3^C\mathbf{e}_1 + m_3[(\boldsymbol{\rho}_{13}^{\mathrm{T}}\boldsymbol{\rho}_{13})(\mathbf{e}_1^T\mathbf{e}_1) - (\mathbf{e}_1^T\boldsymbol{\rho}_{13})^2] \\
&= I_1^{YY} + m_1d_1^2 + \\
&\quad I_2^{XX}S_2^2 + I_2^{YY}C_2^2 + m_2d_2(2a_1S_2 - d_2C_2^2) + \\
&\quad I_3^{XX}S_{23}^2 + I_3^{YY}C_{23}^2 + m_3a_2(2a_1S_2 - a_2C_2^2)
\end{aligned}
\tag{14f}
$$

In order to obtain the scalar terms of eq. (14f), the following transformation matrices were used:

$$
\tilde{\mathbf{Q}}_1 \equiv \mathbf{Q}_1 \equiv \begin{bmatrix} C_1 & 0 & S_1 \\ S_1 & 0 & -C_1 \\ 0 & 1 & 1 \end{bmatrix} \tag{15a}
$$

$$
\tilde{\mathbf{Q}}_2 \equiv \mathbf{Q}_1\mathbf{Q}_2 \equiv \begin{bmatrix} C_1C_2 & -C_1S_2 & S_1 \\ S_1C_2 & -S_1S_2 & -C_1 \\ S_2 & C_2 & 0 \end{bmatrix} \tag{15b}
$$

$$
\tilde{\mathbf{Q}}_3 \equiv \mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3 \equiv \begin{bmatrix} C_1C_{23} & -C_1S_{23} & S_1 \\ S_1C_{23} & -S_1S_{23} & -C_1 \\ S_2 & C_2 & 0 \end{bmatrix} \tag{15c}
$$

where $\mathbf{Q}_i$, for i = 1,2,3, is the orientation matrix between the two successive frames, namely, Frames (i+1) and i, i.e., any vector, $\mathbf{x}$, in Frame (i+1) that is attached to the i^{th} link can be represented in Frame i as, $[\mathbf{x}]_i = \mathbf{Q}_i[\mathbf{x}]_{i+1}$. Moreover, $S_i \equiv Sin\theta_i$; $C_i \equiv Cos\theta_i$, for i = 1,2,3, and $\theta_{23} \equiv \theta_2+\theta_3$. Furthermore, matrix, $\tilde{\mathbf{Q}}_1$, is the transformation matrix from Frame i to the first frame which is fixed to the base of the manipulator. Using the above transformation matrices, a sample calculation to find the scalar, $\mathbf{e}_2^{\mathrm{T}}\mathbf{I}_2^{\mathrm{C}}\mathbf{e}_1$, is shown below:

$$
\begin{aligned}
\mathbf{e}_2^{\mathrm{T}}\mathbf{I}_2^{\mathrm{C}}\mathbf{e}_1 &= [\mathbf{e}_1]_3^{\mathrm{T}}[\mathbf{I}_2^{\mathrm{C}}]_3[\mathbf{e}_1]_3 = [\mathbf{e}_1]_1^{\mathrm{T}}\tilde{\mathbf{Q}}_2^{\mathrm{T}}[\mathbf{I}_2^{\mathrm{C}}]\tilde{\mathbf{Q}}_2[\mathbf{e}_1]_1 \\
&= I_2^{\mathrm{XX}}\mathrm{S}_2^2 + I_2^{\mathrm{YY}}\mathrm{C}_2^2
\end{aligned}
$$

In eq. (14f), the first two terms are constant, where to make the 3rd and 4th terms constant, one should have $I_2^{XX} = I_2^{YY}$. Similarly, the 6th and 7th terms provide, $I_3^{XX} = I_3^{YY}$. Finally, due to the appearance of the joint variables in the 5th and 8th terms they must vanish. Considering their positive parts, the condition, $m_2d_2 + m_3a_2 = 0$, i.e., $m_2d_2 = -m_3a_2$, must hold to make them vanish. Moreover, substituting this condition to the negative parts of the 5th and 8th terms gives, $d_2 = -a_2$.

Based on the Steps 1-6 the parameters of the 3-link arm shown in Fig. 3 are modified to simplify its dynamics. The modified architecture is shown in Fig. 4 whose dynamic equations of motion can be given by

$$\mathbf{I}\,\ddot{\boldsymbol{\theta}} = \boldsymbol{\tau} \tag{16}$$

where the 3×3 GIM, 3-dimensional vectors, $\ddot{\boldsymbol{\theta}}$ and $\boldsymbol{\tau}$, are defined by

$$\mathbf{I} \equiv \begin{bmatrix} i_{11} & 0 & 0 \\ 0 & i_{22} & i_{23} \\ 0 & i_{32} & i_{33} \end{bmatrix};\ \ddot{\boldsymbol{\theta}} \equiv \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix};\ \text{and}\ \boldsymbol{\tau} \equiv \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

in which $\ddot{\theta}_i$, and τ_i, for i=1,2,3, are the joint accelerations, and the joint torques, respectively, whereas the constant elements of the GIM, **I**, are as follows:

$$i_{11} = I_1^{YY} + m_1 d_1^2 + I_2^{XX} + I_3^{XX}$$
$$i_{22} = I_3^{ZZ} + m_3 d_2^2;\ i_{32} = i_{23} = I_3^{ZZ} + m_3 d_2^2;\ \text{and}$$
$$i_{33} = I_3^{ZZ} + m_3 d_3^2$$

in which, $\mathbf{I}_i^{ZZ}$, for i=1,2,3, are the principle mass moment of inertia of the i^{th} link about its Z-axis passing through the mass center, C_i, i.e., $I_i^{ZZ} = m_i a_i^2/12$. Similarly, $\mathbf{I}_i^{XX}$, for i=2,3, is the principle mass moment of inertia of the i^{th} link about its X-axis passing through the mass center, C_i. Moreover, the matrix of the convective inertia terms, i.e., the terms containing the products of the joint rates, **C** of eq. (1), vanishes due to the non-varying elements of the GIM, **I**.

IV. CONCLUSIONS

A dynamic simplification methodology of a serial manipulator based on the usage of the twist propagation matrices and joint-motion propagation vectors, as appeared in eqs. (5a) and (5b), respectively, is presented in this paper. Since the elements of the associated matrices can be explicitly expressed their nature, i.e., constant or zero or varying, can be checked while the manipulator is in motion. It is advantageous from the control point of view if the GIM is constant or constant diagonal. When the GIM is constant diagonal, the control of the robot is decoupled, i.e., each joint can be controlled independently. Constant GIM also guarantees that the matrix of the convective inertia terms will

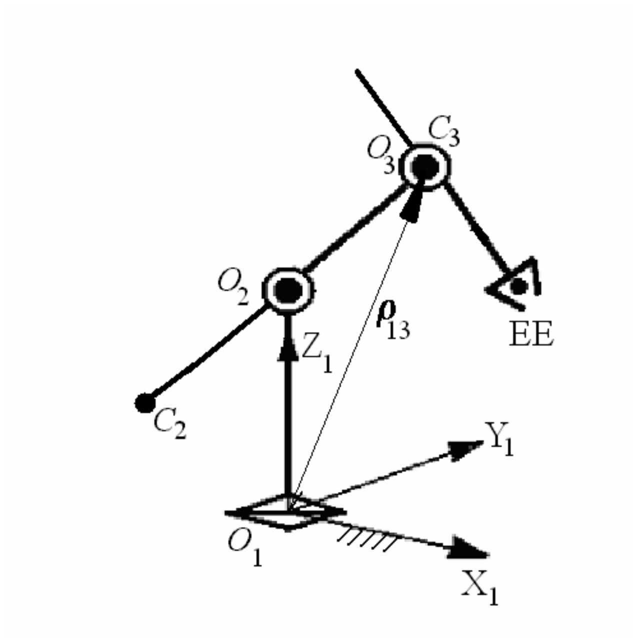

Fig. 4 Modified 3-link arm for simpler dynamics

vanish, thus, simplifying the robot dynamics in the form of eq. (16). Using the proposed methodology kinematic and dynamic parameters of a 3-link spatial manipulator arm are modified for its simpler dynamics given by eq .(16). Future work includes a comprehensive algorithm to choose the parameters for a general n-link serial and closed-loop manipulators with arbitrary architecture.

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