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An optimization technique for the balancing of spatial mechanisms

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Abstract

This paper presents a new generic optimization technique for the balancing of the shaking force and shaking moment due to inertia forces in spatial mechanisms. For given dimensions and input speed of a mechanism, the inertia forces depend only upon the mass distribution of the moving links. The equimomental system of seven point-masses is introduced to represent the inertial properties of the links and to identify optimizing variables. The effectiveness of the proposed methodology is illustrated using a spatial RSSR mechanism.

Keywords: Mechanism balancing; Shaking force; Shaking moment; Equimomental system; RSSR mechanism

1. Introduction

Balancing of shaking force and shaking moment in high speed mechanisms/machines reduces the forces transmitted to the frame. In effect, this minimizes the noise and wear, and improves the performance of a mechanism. The balancing of shaking force has been studied by various researchers [1–15], and others. A considerable amount of research on balancing of shaking force and shaking moment in planar mechanisms has been carried out in the past [1–5]. In contrast to rapid progress in balancing theory and techniques for planar mechanisms, the understanding of shaking force and shaking moment balancing of spatial mechanisms is very limited. Kaufman and Sandor [6] presented a complete force balancing of spatial mechanisms like (revolute–spherical–spherical–revolute) RSSR and (revolute–spherical–spherical–prismatic) RSSP. Their approaches are based on the generalization of the planar balancing theory developed by Berkof and Lowen [5], a technique of linearly independent vectors. Using the real vectors and the concept of retaining the stationary centre of total mass, Bagci [7] has obtained the design equations for force balancing of various mechanisms, whereas Ning-Xin Chen [8,9] extended the concept of linearly independent vectors to single loop spatial n-bar linkages with some restricted kinematic pairs for the derivation of the equations of complete shaking force balancing. In

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addition to the balancing of shaking force, shaking moment balancing can be achieved by adding the dyads [10,11] and rotating mass balancers [12]. However, these balancing methods use counterweights and/or dyads that are restricted to few specific mechanisms only.

Shaking force balancing of a mechanism does not imply that the shaking moment is also balanced. In fact, shaking force balancing increases unbalancing of shaking moment and other dynamic quantities. Hence, balancing a mechanism requires a trade-off between the shaking force, shaking moment, and other quantities, say the input torque. Although combined balancing of the shaking force and shaking moment is more useful but it is difficult one. Simultaneous minimization of shaking force, shaking moment, and other quantities using the dynamical equivalent system of point-masses and optimum mass distribution has been attempted in [13,14]. However, the results do not show significant improvement in the performances. Moreover, the techniques like genetic algorithm etc. were also applied to the optimum balancing of shaking force and shaking moment for the spatial RSSR mechanism [15]. Note that the concept of equimomental system of point-masses [16–18] and its effective use in the spatial mechanism balancing have not been reported so far, which may be attributed to the formulation difficulty of dynamic equations of motion, and the nonlinear nature of the equimomental conditions.

In this paper, balancing of spatial mechanisms based on the concept of equimomental systems of point-masses is proposed. The balancing problem is formulated as an optimization problem. The dynamic equations of motion of mechanism are formulated in such a way that they can be converted easily to a set of those that corresponds to the equimomental point-masses. A set of seven point-masses to represent rigid link is proposed here to convert the nonlinear equimomental conditions into the linear ones. The design variables and corresponding constraints are identified from the parameters of equimomental point-masses. The solution to the optimization problem then redistributes the link masses so that the combined shaking force and shaking moment is minimum value. The proposed methodology is illustrated with a spatial RSSR mechanism.

This paper is organized as follows. Section 2 explains the concept of equimomental system for a rigid link moving in the three-dimensional cartesian space. Optimization problem is then formulated in Section 3 for the mechanism balancing. Using the spatial RSSR mechanism, the effectiveness of the methodology is illustrated in Section 4. Finally, conclusions are given in Section 5.

2. Equimomental system

In this section, it is shown how to represent a rigid body as an equimomental system of a set of point masses. A point mass is defined as a mass concentrates at a point. A rigid body and a system of point masses are said to be dynamically equivalent or more specifically equimomental if they have same total mass, the same centre of mass, and the same inertia with respect to the same coordinate frame [16].

Let us consider a three-dimensional rigid body, Fig. 1, of total mass m, position of the mass centre $(\bar{x}, \bar{y}, \bar{z})$, the moments of inertia I_{xx}, I_{yy}, I_{zz}, and the products of inertia I_{xy}, I_{yz}, I_{zx} referred to a body fixed frame $OXYZ$.

Let there be n point-masses, m_i, which are rigidly fixed to the frame at positions (x_i, y_i, z_i), for $i = 1, \ldots, n$. The system of n point-masses will be dynamically equivalent to the rigid body if the following conditions are satisfied:

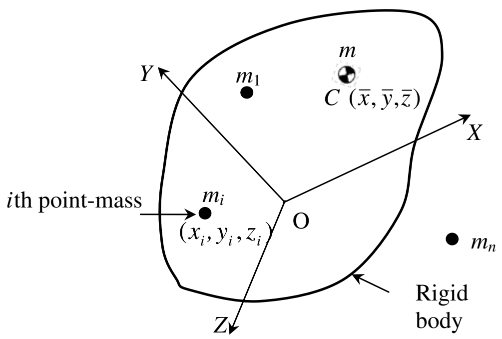

Fig. 1. Dynamic equivalence of n point-masses for a rigid body.

$$\sum_{i=1}^{n} m_i = m \tag{1}$$

$$\sum_{i=1}^{n} m_i x_i = m\bar{x}; \quad \sum_{i=1}^{n} m_i y_i = m\bar{y}; \quad \sum_{i=1}^{n} m_i z_i = m\bar{z} \tag{2–4}$$

$$\sum_{i=1}^{n} m_i x_i y_i = I_{xy}; \quad \sum_{i=1}^{n} m_i y_i z_i = I_{yz}; \quad \sum_{i=1}^{n} m_i z_i x_i = I_{zx} \tag{5–7}$$

$$\sum_{i=1}^{n} m_i (y_i^2 + z_i^2) = I_{xx}; \quad \sum_{i=1}^{n} m_i (z_i^2 + x_i^2) = I_{yy}; \quad \sum_{i=1}^{n} m_i (x_i^2 + y_i^2) = I_{zz} \tag{8–10}$$

Eq. (1) ensures that the total mass of the equimomental points is equal to the mass of the body. Eqs. (2)–(4) satisfy conditions of the mass centre location, whereas Eqs. (5)–(10) ensure the same inertia tensor for the equivalent point mass system and the original rigid body about point O. Since each point-mass is identified with four parameters, namely, m_i, x_i, y_i, z_i, the set of n point-masses requires $4n$ parameters to describe them completely. These parameters, however, must satisfy ten conditions given by Eqs. (1)–(10). Hence, the smallest positive integer n that will provide to satisfy the ten constraints is $4n \geqslant 10$, i.e., 3. Note that the three points determine a plane. As a result, they are not sufficient to describe a rigid body motion in space, and a minimum of four point-masses are required to determine the equimomental system of point-masses, as also reported in [17].

Note that Eqs. (2)–(10), are 2nd and 3rd order polynomials in m_i, x_i, y_i, and z_i. In case coordinates x_i, y_i, z_i, are specified for all the n point-masses, the equations are turn out to be linear in unknown, m_i. However, the number of unknowns is less than the number of equations, Eqs. (1)–(10), for $n < 10$ and vice-versa. Eqs. (1)–(10) have infinite solution. Hence, linearization of the equimomental conditions is done using the special form of the moment of inertia conditions, Eqs. (8)–(10). If one chooses the absolute values of the coordinates, x_i, y_i, and z_i as h_x, h_y, and h_z, respectively, for all the n points, then from Eqs. (8)–(10)

$$h_x^2 = \frac{-I_{xx} + I_{yy} + I_{zz}}{2m}; \quad h_y^2 = \frac{I_{xx} - I_{yy} + I_{zz}}{2m}; \quad h_x^2 = \frac{I_{xx} + I_{yy} - I_{zz}}{2m} \tag{11–13}$$

where Eq. (1) is used to obtain Eqs. (11)–(13). The moments of inertia, I_{xx}, I_{yy}, and I_{zz}, are such that the sum of any two of them is always greater than the third one [16], which implies that

$$(-I_{xx} + I_{yy} + I_{zz}) > 0; \quad (I_{xx} - I_{yy} + I_{zz}) > 0; \quad \text{and} \quad (I_{xx} + I_{yy} - I_{zz}) > 0 \tag{14}$$

Therefore, h_x, h_y, and h_z never have imaginary values. Knowing the coordinates of point-masses, the remaining unknown masses, namely, $m_1, \ldots, m_n$, can be solved uniquely from the remaining seven linear algebraic equality, Eqs. (1)–(7), if $n = 7$. Hence, such a set of seven point-masses is the suitable to represent a rigid body, as the number of unknowns is equal to the number of equations. Moreover, to avoid the coincidence of two or more points, they must be placed at unique locations. This is only possible if all the points have unique coordinates. Since the positive and negative values of h_x, h_y, and h_z represent Cartesian coordinates of the point-masses, they form a rectangular parallelepiped whose centre is at origin point O, and the sides are $2h_x$, $2h_y$ and $2h_z$, as shown in Fig. 2.

On substitution of the coordinates of the seven point-masses, the dynamically equivalent conditions, Eqs. (1)–(7), can be written in the compact form as

$$\mathbf{Km} = \mathbf{b} \tag{15}$$

where the 7-vectors, $\mathbf{m}$ and $\mathbf{b}$, and the 7×7 matrix, $\mathbf{K}$, are defined as

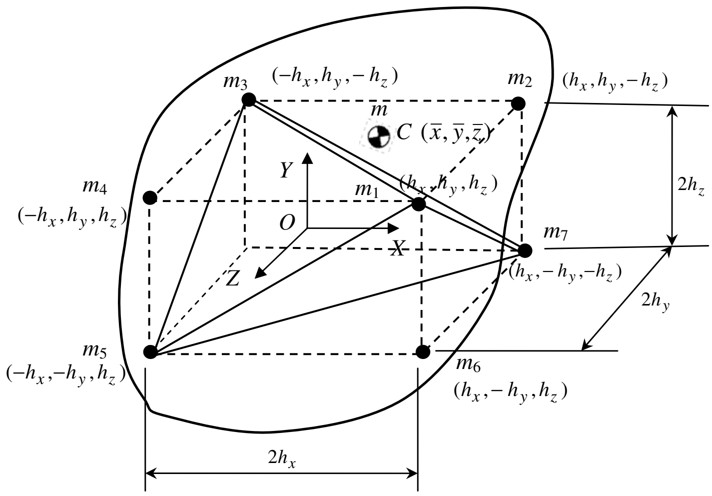

Fig. 2. Equimomental system of seven point-masses for a rigid body.

$$\mathbf{m} \equiv [m_1 \quad m_2 \quad m_3 \quad m_4 \quad m_5 \quad m_6 \quad m_7]^{\mathrm{T}} \tag{16}$$

$$\mathbf{b} \equiv \left[m \quad \frac{m\bar{x}}{h_x} \quad \frac{m\bar{y}}{h_y} \quad \frac{m\bar{z}}{h_z} \quad \frac{I_{xy}}{h_x h_y} \quad \frac{I_{yz}}{h_y h_z} \quad \frac{I_{zx}}{h_z h_x} \right]^{\mathrm{T}} \tag{17}$$

$$K \equiv \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 \end{bmatrix} \tag{18}$$

The vector of unknowns, **m**, is then easily solved as

$$\mathbf{m} = \mathbf{K}^{-1}\mathbf{b} \tag{19}$$

where $\mathbf{K}^{-1}$ is given by

$$\mathbf{K}^{-1} \equiv \begin{bmatrix} 0.25 & 0 & 0 & 0 & 0.25 & 0.25 & 0.25 \\ 0 & 0.25 & 0.25 & 0 & 0 & -0.25 & -0.25 \\ 0.25 & -0.25 & 0 & -0.25 & 0 & 0 & 0.25 \\ 0 & 0 & 0.25 & 0.25 & -0.25 & 0 & -0.25 \\ 0.25 & -0.25 & -0.25 & -0 & 0.25 & 0 & 0 \\ 0 & 0.25 & 0 & 0.25 & -0.25 & -0.25 & 0 \\ 0.25 & 0 & -0.25 & -0.25 & 0 & 0.25 & 0 \end{bmatrix} \tag{20}$$

Now, if the origin of the frame, O, coincides with the mass centre, C, and the axes of the frame are along the principal axes, i.e., $\bar{x} = \bar{y} = \bar{z} = 0$ and $I_{xy} = I_{yz} = I_{zx} = 0$, it is evident from Eq. (19) that $m_1 = m_3 = m_5 = m_7 = 0.25m$, and $m_2 = m_4 = m_6 = 0$, irrespective of the values of h_x, h_y and h_z. Therefore, the four point masses, m_1, m_3, m_5, and m_7, with equal masses are found. These point-masses form a tetrahedron, and each point lies at the vertex of the tetrahedron shown in Fig. 2. This tetrahedron may be regarded as inscribed within a sphere whose radius is $\sqrt{h_x^2 + h_y^2 + h_z^2}$. These results conform to those reported by Routh [16], i.e., "*Four particles of equal mass can always be found which are equimomental to any given solid body*". However, no proof or justification was provided in [16], which is reported here for the first time. Also it is confirmed with [18] that the equimomental system of point-masses of a rigid body form a tetrahedron with four equal point-masses.

3. Formulation of the balancing problem

As pointed out in Section 1, the balancing problem is to find mass, location of the mass centre, and inertia of each link of a mechanism whose dimensions and input motion are given so that shaking forces transmitted to the frame are minimum. The problem is treated as an optimization problem, which is formulated in this section. To do so, first the Newton–Euler (NE) equations of motion are derived in the parameters, identified as the design variables, of the point-masses.

3.1. Coordinate systems

In order to specify the configuration of a spatial mechanism, body-fixed coordinate frames are defined as shown in Fig. 3. It is assumed that each link is coupled to its previous and the next one with one degree-of-freedom (dof) joints. The ith joint couples the $(i-1)$st link with the ith one. With each link, namely, the $(i-1)$st one, a Cartesian coordinate system, $O_iX_iY_iZ_i$, denoted by $\mathscr{F}_i$, is attached at O_i. The dimensions and configuration of the linkage are then determined by set of parameters, referred as Denavit–Hartenberg (DH) parameters [20,30], a_i, b_i, α_i, and θ_i, as shown in Fig. 3. Correspondingly, the transformations of vectors and matrices between the frames $\mathscr{F}_{i+1}$ and $\mathscr{F}_i$ can be easily obtained. In this paper, representation of vectors and matrices in a frame, say, $\mathscr{F}$, is denoted with $[.]_{\mathscr{F}}$, where '.' is the vector or the matrix.

3.2. Dynamic equations of motion

Referring to the ith link of a spatial mechanism, Fig. 4, O_i and O_{i+1} are the origins of the coordinate frames, $O_iX_iY_iZ_i$ ($\mathscr{F}_i$) and $O_{i+1}X_{i+1}Y_{i+1}Z_{i+1}$ ($\mathscr{F}_{i+1}$), respectively, where the frame, $\mathscr{F}_{i+1}$, is fixed to the ith link. To express the rigid body kinematics and dynamics, two 6-vectors *twist* and *wrench*, which are analogous to the velocity and force of a particle, of the ith body are defined as

$$\mathbf{t}_i \equiv \begin{bmatrix} \boldsymbol{\omega}_i \\ \mathbf{v}_i \end{bmatrix} \quad \text{and} \quad \mathbf{w}_i \equiv \begin{bmatrix} \mathbf{n}_i \\ \mathbf{f}_i \end{bmatrix} \tag{21}$$

where 3-vectors $\boldsymbol{\omega}_i$ and $\mathbf{v}_i$ are the angular velocity of the ith body and the linear velocity of point O_i on the body, respectively. Accordingly, 3-vectors $\mathbf{n}_i$ and $\mathbf{f}_i$ are the resultant moment including those due to reaction

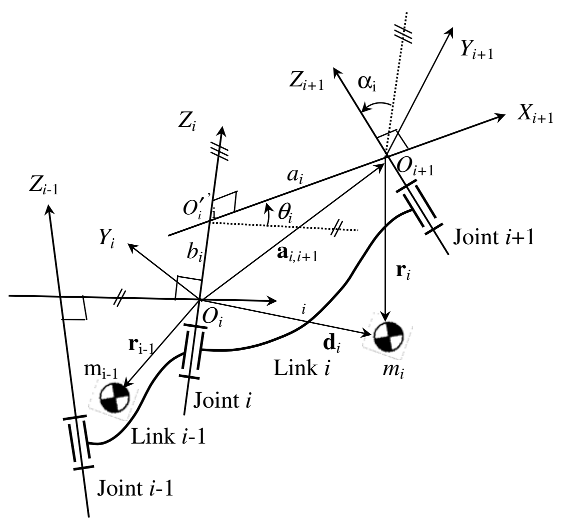

Fig. 3. Coordinate frames and associated parameters.

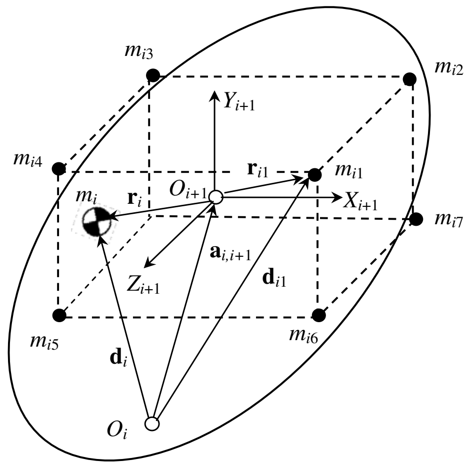

Fig. 4. Equimomental point-masses and their locations for ith link.

forces on the body about O_i, and the resultant force at O_i, respectively. Now the Newton–Euler (NE) equations of motion [22] are expressed as

$$\mathbf{M}_i\dot{\mathbf{t}}_i + \mathbf{W}_i\mathbf{M}_i\mathbf{E}_i\mathbf{t}_i = \mathbf{w}_i \tag{22}$$

where the 6×6 matrices $\mathbf{M}_i$, $\mathbf{W}_i$ and $\mathbf{E}_i$ are the mass, angular velocity, and coupling matrix, respectively, and defined as

$$\mathbf{M}_i \equiv \begin{bmatrix} \mathbf{I}_i & m_i\tilde{\mathbf{d}}_i \\ -m_i\tilde{\mathbf{d}}_i & m_i\mathbf{1} \end{bmatrix}; \quad \mathbf{W}_i \equiv \begin{bmatrix} \tilde{\boldsymbol{\omega}}_i & \mathbf{O} \\ \mathbf{O} & \tilde{\boldsymbol{\omega}}_i \end{bmatrix}; \quad \text{and} \quad \mathbf{E}_i \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \tag{23}$$

In which $\mathbf{I}_i$ is the inertia tensor about O_i. $\tilde{\mathbf{d}}_i$ and $\tilde{\boldsymbol{\omega}}_i$ are the 3×3 skew symmetric matrices corresponding to vectors $\mathbf{d}_i$ and $\boldsymbol{\omega}_i$, respectively, i.e., $\tilde{\mathbf{d}}_i\mathbf{x} = \mathbf{d}_i \times \mathbf{x}$, and $\tilde{\boldsymbol{\omega}}_i\mathbf{x} = \boldsymbol{\omega}_i \times \mathbf{x}$ for the 3-vector, $\mathbf{x}$. Moreover, $\mathbf{1}$ and $\mathbf{O}$ are the 3×3 identity and zero matrices. In order to represent the mass matrix, $\mathbf{M}_i$ of Eq. (23), in terms of the parameters of point-masses, the ith rigid link is modelled as seven equimomental point-mass system as presented in Section 2. Referring to Fig. 4, the point-masses are placed at the corners of a rectangular parallelepiped whose centre is point O_{i+1}, and the sides are $2h_{ix}, 2h_{ix}, 2h_{iz}$ along the axes, X_{i+1}, Y_{i+1} and Z_{i+1}, respectively. The point-masses, $m_{i1}, \ldots, m_{i7}$, are fixed in the local frame, $O_{i+1}X_{i+1}Y_{i+1}Z_{i+1}$, that is attached to the ith body. The 3-vectors, $\mathbf{d}_{ij}$ and $\mathbf{r}_{ij}$, are defined from the origins O_i and O_{i+1} to the point-mass, m_{ij}, respectively. Subscripts i and j denote ith link and jth point-mass. The components of the vectors, $\mathbf{r}_{ij}$, in the body fixed frame, $O_{i+1}X_{i+1}Y_{i+1}Z_{i+1}$, are then given in Table 1.

Now, using the equimomental conditions, Eqs. (2)–(4), one can express vector $\mathbf{d}_i$ in terms of $\mathbf{d}_{ij}$'s representing the positions of point masses from O_i, i.e.,

$$\mathbf{d}_i = \frac{1}{m_i}\sum_{j=1}^{7} m_{ij}\mathbf{d}_{ij} \tag{24}$$

where $\mathbf{d}_{ij} \equiv \mathbf{a}_{i,i+1} + \mathbf{r}_{ij}$, and the mass of the body is

$$m_i = \sum_{j=1}^{7} m_{ij} \tag{25}$$

Table 1
Components of vectors $\mathbf{r}_{ij}$ in the $(i+1)$st frame

	$[\mathbf{r}_{i1}]_{i+1}$	$[\mathbf{r}_{i2}]_{i+1}$	$[\mathbf{r}_{i3}]_{i+1}$	$[\mathbf{r}_{i4}]_{i+1}$	$[\mathbf{r}_{i5}]_{i+1}$	$[\mathbf{r}_{i6}]_{i+1}$	$[\mathbf{r}_{i7}]_{i+1}$
Along X_{i+1}	h_{ix}	h_{ix}	$-h_{ix}$	$-h_{ix}$	$-h_{ix}$	h_{ix}	h_{ix}
Along Y_{i+1}	h_{iy}	h_{iy}	h_{iy}	h_{iy}	$-h_{iy}$	$-h_{iy}$	$-h_{iy}$
Along Z_{i+1}	h_{iz}	$-h_{iz}$	$-h_{iz}$	h_{iz}	h_{iz}	h_{iz}	$-h_{iz}$

Denoting $\mathbf{d}_{ij} \equiv [d_{ijx}, d_{ijy}, d_{ijz}]^{\mathrm{T}}$ and using Eqs. (5)–(10), the inertia tensor, $\mathbf{I}_i$, about the origin, O_i, as in Eq. (23), in terms of the point mass parameters has following representation:

$$\mathbf{I}_i = \begin{bmatrix} \sum_{j=1}^{7} m_{ij}(d_{ijy}^2 + d_{ijz}^2) & -\sum_{j=1}^{7} m_{ij} d_{ijx} d_{ijy} & -\sum_{j=1}^{7} m_{ij} d_{ijx} d_{ijz} \\ & \sum_{j=1}^{7} m_{ij}(d_{ijz}^2 + d_{ijx}^2) & -\sum_{j=1}^{7} m_{ij} d_{ijy} d_{ijz} \\ \text{Sym} & & \sum_{j=1}^{7} m_{ij}(d_{ijx}^2 + d_{ijy}^2) \end{bmatrix} \tag{26}$$

Eqs. (24)–(26) define the mass matrix, $\mathbf{M}_i$, of the ith link in terms of the parameters of equimomental seven point masses. Now, to compute the joint reactions and driving torques/forces the dynamic equations for the whole system can be derived as proposed in [22,29] and others [19,21]. Knowing the joint reactions and the driving torques/forces the shaking force and shaking moment are determined in the next section.

3.3. Shaking force and shaking moment

Shaking force is defined as the reaction of the resultant inertia forces, whereas shaking moment about any particular point is the reactions of the resultant inertia couples and the moment of the inertia forces about that point [23]. Using these definitions, the shaking force and shaking moment with respect to O_1 in a mechanism having n moving links are obtained as

$$\mathbf{f}_{\mathrm{sh}} = -\sum_{i=1}^{n} \mathbf{f}_i^*; \quad \text{and} \quad \mathbf{n}_{\mathrm{sh}} = -\sum_{i=1}^{n} (\mathbf{n}_i^* + \tilde{\mathbf{a}}_{1,i} \mathbf{f}_i^*) \tag{27}$$

where $\mathbf{f}_i^*$ and $\mathbf{n}_i^*$ are 3-vectors of inertia force and inertia moment of the ith body acting at and about origin O_i, respectively, and are the vector components of $\mathbf{w}_i^*$ similar to definition of Eq. (21). The 3×3 matrix $\tilde{\mathbf{a}}_{1,i}$ is the skew symmetric matrix corresponding to the 3-vector, $\mathbf{a}_{1,i}$, pointing O_i from O_1. The point O_1 is the origin of the frame $X_1 Y_1 Z_1$ attached to the fixed link. It is evident from Eq. (27) that the shaking moment is pure torque for fully force balanced mechanism. Referring to Fig. 5, the equilibrium of forces and moments are expressed as

$$\mathbf{f}_i^* = \mathbf{f}_i^{\mathrm{E}} + \mathbf{f}_{i-1,i} - \mathbf{f}_{i,i+1} \tag{28}$$

$$\mathbf{n}_i^* = \mathbf{n}_i^{\mathrm{E}} + \mathbf{n}_{i-1,i} - \mathbf{n}_{i,i+1} + \tilde{\mathbf{a}}_{i,i+1} \mathbf{f}_{i,i+1} \tag{29}$$

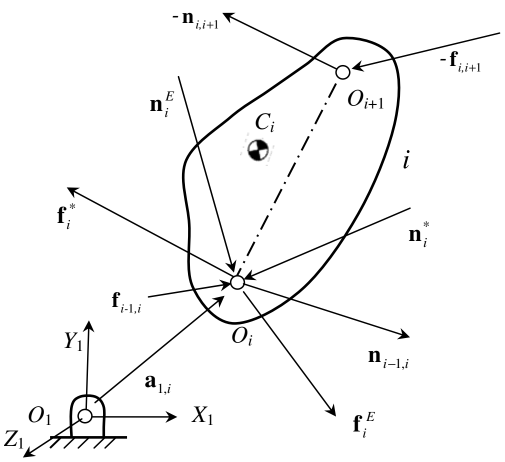

Fig. 5. Free body diagram of the ith link.

where 3-vectors, $\mathbf{f}_{i-1,i}, \mathbf{n}_{i-1,i}$, and $\mathbf{f}_{i,i+1}, \mathbf{n}_{i,i+1}$, are the constraint forces and moments at joints, O_i and O_{i+1}, respectively, whereas 3-vectors, $\mathbf{f}_i^{\mathrm{E}}$ and $\mathbf{n}_i^{\mathrm{E}}$, are external force and torque acting on the ith body at and about O_i, respectively.

Let us assume that links, $k = 1, \ldots, p$, are connected to the fixed link $(n + 1)$. Using Eqs. (28)–(29), the shaking force and shaking moment expressions, Eq. (27), are then rewritten as

$$\mathbf{f}_{\mathrm{sh}} = -\sum_{k=1}^{p} \mathbf{f}_{n+1,k} - \sum_{i=1}^{n} \mathbf{f}_i^{\mathrm{E}} \tag{30}$$

$$\mathbf{n}_{\mathrm{sh}} = -\sum_{k=1}^{p} (\mathbf{n}_{n+1,k} + \tilde{\mathbf{a}}_{1,j}\mathbf{f}_{n+1,k}) - \sum_{i=1}^{n} (\mathbf{n}_i^{\mathrm{E}} + \tilde{\mathbf{a}}_{1,i}\mathbf{f}_i^{\mathrm{E}}) \tag{31}$$

where $\mathbf{f}_{n+1,k}$ represents the reaction force of the fixed link, $(n + 1)$, on the kth link connected to it. For all the other links $(p < i \leqslant n)$ that are not connected to the fixed one, the term $\mathbf{f}_{n+1,k}$ vanishes.

3.4. Optimization problem

In this section, two optimization techniques are developed to reduce shaking force and shaking moment: (1) redistribution of the mass of moving links and (2) counterweighting the moving links.

3.4.1. Mass redistribution method

The shaking force and shaking moment are the resultant of the inertia forces and moments of moving links. When the dimensions and the input speed of a mechanism are given, the inertia forces depend only upon the mass distribution of the moving links. Hence, mass redistribution is the obvious choice to balance mechanisms. First the inertial properties of the links are represented using the seven point-mass model. The seven point masses of each link, $m_{i1}, \ldots, m_{i7}$, which are located at the corners of a rectangular parallelepiped of sides, $2h_{ix}, 2h_{ix}$, and $2h_{iz}$ as explained in Section 2, are taken as the design variables. The values of h_{ix}, h_{ix}, and h_{iz} are calculated from the moments of inertia of each original link using Eqs. (11)–(13). For a mechanism having n moving links, a $7n$-vector of the design variables, $\mathbf{x}$, is then defined as

$$\mathbf{x} \equiv [\mathbf{m}_1^{\mathrm{T}}, \ldots, \mathbf{m}_n^{\mathrm{T}}]^{\mathrm{T}} \tag{32}$$

where the 7-vectors $\mathbf{m}_i$ is defined similar to $\mathbf{m}$ in Eq. (16) as

$$\mathbf{m}_i \equiv [\, m_{i1} \quad m_{i2} \quad m_{i3} \quad m_{i4} \quad m_{i5} \quad m_{i6} \quad m_{i7} \,]^{\mathrm{T}}$$

There are many possible criteria by which the shaking force and shaking moment transmitted to the fixed link of the mechanism can be minimized. For example, one criterion could be the root mean squares (RMS) of shaking force, shaking moment, and required input-torque for a given motion, and/or combination of these. Besides RMS values, there are other ways to specify the dynamic quantities, namely, by maximum values, or by the amplitude of the specified harmonics, or by the amplitudes at certain point during the motion cycle. The choice of course depends on the requirements. Here, the RMS value is preferred over others as it gives equal emphasis on the results of every time instances of the cycle and every harmonic component. The RMS values of the normalized shaking force, $\bar{f}_{\mathrm{sh}}$, and the normalized shaking moment, $\bar{n}_{\mathrm{sh}}$, at p discrete positions of the mechanism are defined as

$$\tilde{f}_{\mathrm{sh}} \equiv \frac{1}{p}\sqrt{\sum \bar{f}_{\mathrm{sh}}^2} \quad \text{and} \quad \tilde{n}_{\mathrm{sh}} \equiv \frac{1}{p}\sqrt{\sum \bar{n}_{\mathrm{sh}}^2} \tag{33}$$

where $\tilde{f}_{\mathrm{sh}}$ and $\tilde{n}_{\mathrm{sh}}$ are the RMS values of the normalized shaking force and the normalized shaking moment, respectively. Now, optimality criteria can be defined as the weighted sum of the competing dynamic quantities, namely, shaking force, shaking moment, input torque, and the reactions due to the frame of mechanism. However, it is obvious from Eqs. (30)–(31) that the shaking force and shaking moment include the frame reactions and the input torque, respectively. Hence, it is sufficient to form optimality criteria as weighted sum of the shaking force and shaking moment. Taking the root mean square (RMS) values of the normalized shaking force, $\bar{f}_{\mathrm{sh}}$, and the normalized shaking moment, $\bar{n}_{\mathrm{sh}}$, an optimality criterion is proposed next as

$$z = w_1\tilde{f}_{sh} + w_2\tilde{n}_{sh} \tag{34}$$

where w_1 and w_2 are the weighting factors whose values may vary depending on an application and $\tilde{f}_{sh}$ and $\tilde{n}_{sh}$ are the RMS values of the shaking force and shaking moment. Considering the lower and upper limits on the link masses and their mass centre locations, the problem of mechanism balancing is finally stated as

$$\underset{\text{w.r.t.}x}{\text{Minimize}} \quad z(\mathbf{x}) = w_1\tilde{f}_{sh} + w_2\tilde{n}_{sh} \tag{35a}$$

$$\text{Subject to} \quad m_{i,\min} \leqslant \sum_{j=1}^{7} m_{ij} \leqslant m_{i,\max} \tag{35b}$$

$$r_{i,\min} \leqslant r_i \leqslant r_{i,\max} \tag{35c}$$

for $i = 1, \ldots, n$, whereas $m_{i,\min}$ and $m_{i,\max}$, and $r_{i,\min}$ and $r_{i,\max}$ are the minimum and maximum limits on the mass and its mass centre location of the ith link. Note here that the minimum moments of inertia of the ith link are only depend on the $m_{i,\min}$. For example, $I_{ixx,\min}$ with respect to O_i in the link-fixed frame is given by

$$I_{ixx,\min} = \sum_{j=1}^{7} m_{ij}(d_{ijy}^2 + d_{ijz}^2) = m_{i,\min}[(a_{i,i+1y} + r_{ijy})^2 + (a_{i,i+1z} + r_{ijz})^2] \tag{36}$$

where $r_{ijx}, r_{ijy}, r_{ijz}$ are components of the 3-vector $\mathbf{r}_{ij}$ given in Table 1. Moreover, $a_{i,i+1x}, a_{i,i+1y}, a_{i,i+1z}$ are the components of the link length vector, $\mathbf{a}_{i,i+1}$. Therefore, the moments of inertia of the links are governed by the bound chosen on the link masses, as $\mathbf{r}_{ij}$ and $\mathbf{a}_{i,i+1}$ are constants in the local coordinate frame. Hence, the optimization problem finds a value of each point mass of each link while the total mass and its mass centre location of each link are subjected to lower and upper limits. From the optimized values of point-masses m_{ij}^*, optimized total mass m_i^*, location of the mass centre $(\bar{x}_i^*, \bar{y}_i^*, \bar{z}_i^*)$, and inertia $I_{i,xx}^*, I_{i,yy}^*, I_{i,zz}^*, I_{i,xy}^*, I_{i,yz}^*, I_{i,zx}^*$ of each link are determined using the equimomental conditions, Eqs. (1)–(10).

3.4.2. Counterweight method

When the given unbalanced mechanism has been kinematically synthesized, and the mass distribution of the links has been determined according to load bearing capacity, etc. the mechanism can be balanced by attaching counterweights to the moving links. Assume that the counterweight mass, m_i^b, is attached to the ith link at $(\bar{x}_i^b, \bar{y}_i^b, \bar{z}_i^b)$, as shown in Fig. 5. The equimomental system of the resulting link is shown in Fig. 6, where it is assumed that point-masses, m_{ij}^b, are placed at the same locations where the point-masses of the original link, m_{ij}^o were located. Then the mass of the counterweight m_i^b, its mass centre location $\mathbf{r}_i^b$, and its inertias $I_{i,xx}^b, I_{i,yy}^b I_{i,zz}^b, I_{i,xy}^b, I_{i,yz}^b$, and $I_{i,zx}^b$ can be obtained using the equimomental conditions, Eqs. (1)–(10). Now, for a mechanism having n moving links, the $7n$-vector of the design variables, $\mathbf{x}^b$, is

$$\mathbf{x}^b \equiv [\mathbf{m}_1^{b\mathrm{T}}, \ldots, \mathbf{m}_n^{b\mathrm{T}}]^\mathrm{T} \tag{37}$$

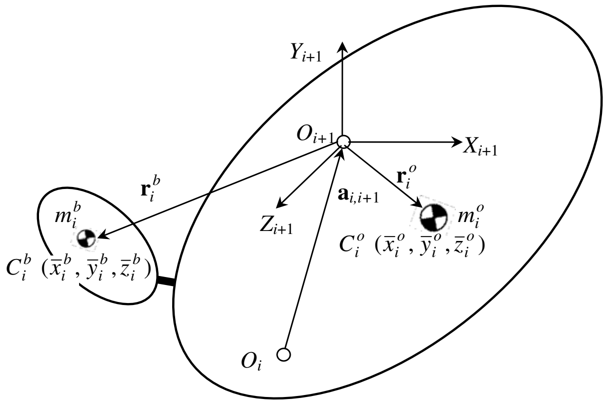

Fig. 6. The ith link with counterweight.

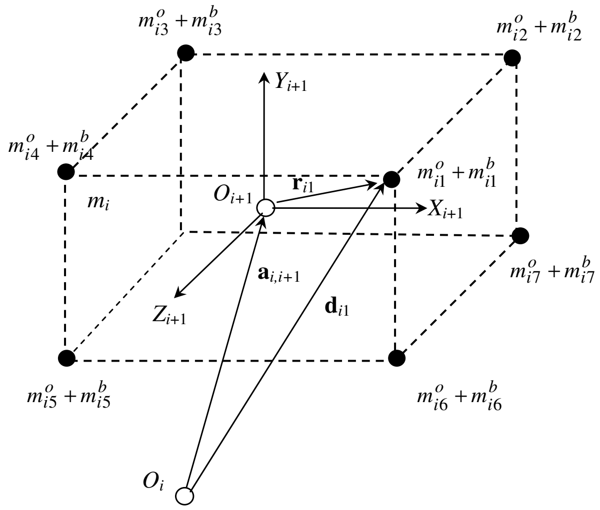

Fig. 7. The equimomental system of point-masses for the counterweighted ith link.

where the 7-vectors, $\mathbf{m}_i^b$, is as follows:

$$\mathbf{m}_i^b \equiv [m_{i1}^b \quad m_{i2}^b \quad m_{i3}^b \quad m_{i4}^b \quad m_{i5}^b \quad m_{i6}^b \quad m_{i7}^b]^{\mathrm{T}}$$

Considering the lower and upper limits on the counterweight masses and their mass centre locations the balancing problem is stated similar to Eq. (35) as

$$\underset{\text{w.r.t.}\mathbf{x}^b}{\text{Minimize}} \quad z(\mathbf{x}^b) = w_1\tilde{f}_{\text{sh}} + w_2\tilde{n}_{\text{sh}} \tag{38a}$$

$$\text{Subject to} \quad m_{i,\min}^b \leqslant \sum_{j=1}^{7} m_{ij}^b \leqslant m_{i,\max}^b \tag{38b}$$

$$r_{i,\min}^b \leqslant r_i^b \leqslant r_{i,\max}^b) \tag{38c}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, 7$ (see Fig. 7).

4. Application to the RSSR mechanism

The RSSR mechanism, Fig. 8, is kinematically equivalent to the R (RRR) (RR) R mechanism [24] as shown in Fig. 9. The (RR) and (RRR) joint arrangements represent two and three mutually orthogonal axes of the revolute pairs intersecting at a point, respectively. It is fact that rotation of coupler link about its axis is redundant dof and does not affect the overall motion of the mechanism. The redundant dof is removed in $7R$ mechanism. The $7R$ mechanism has one degree of freedom and the joint 7 is assumed be driven by an actuator.

The links in Fig. 9 are numbered consecutively as $1, \ldots, 7, -7$ being the fixed link. The joints are also numbered so that joint i connects link i and $i+1$. The DH notations defined in Section 3.1 are now

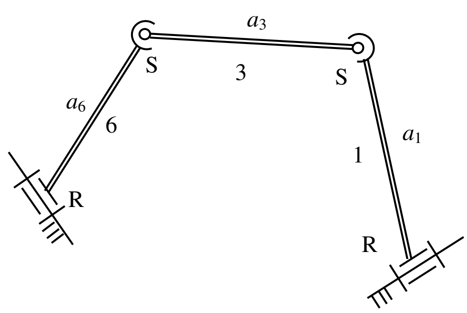

Fig. 8. The RSSR mechanism.

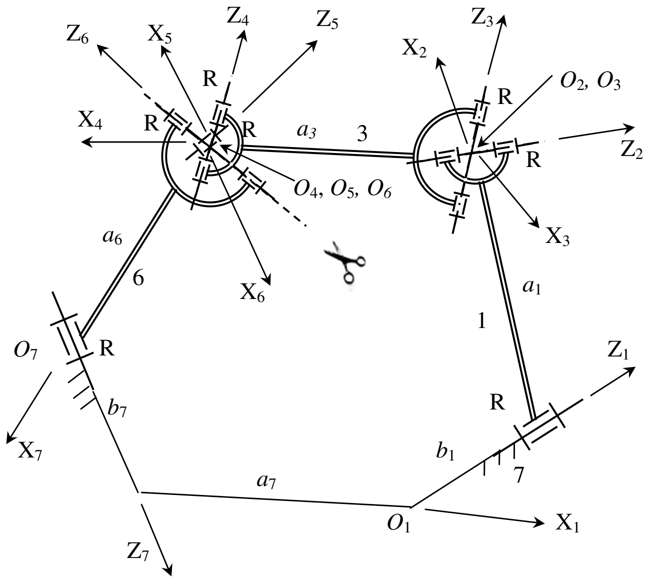

Fig. 9. The R (RRR) (RR) R mechanism.

used to define the architecture of the 7R mechanism which is kinematically equivalent to the RSSR mechanism if

$$a_2 = a_4 = a_5 = 0; \quad b_2 = b_3 = b_4 = b_5 = b_6 = 0; \quad \alpha_2 = \alpha_4 = \alpha_5 = 0 \tag{39}$$

The 3-vectors, $\mathbf{a}_{i,i+1}$, as shown in Fig. 3, is then given by

$$\mathbf{a}_{i,i+1} = b_i\mathbf{e}_i + a_i\mathbf{x}_{i+1} \tag{40}$$

where $\mathbf{e}_i$ and $\mathbf{x}_{i+1}$ are the unit 3-vectors along the axes Z_i and X_{i+1}, respectively. The unit vectors have simple form in ith and (i+1)st coordinate frame, namely, $[\mathbf{e}_i]_i = [0, 0, 1]^T$ and $[\mathbf{x}_{i+1}]_{i+1} = [1, 0, 0]^T$. The closed-loop mechanism is now made open by cutting the joint 6. The cutting line is indicated in Figs. 9 and 11 by dashed line along the axis of joint 6. The resulting two open subsystems: subsystem I with one moving link 6, and subsystem II with five serially connected moving links, $1, \ldots, 5$. Both the subsystems connected to the fixed link 7.

The loop-closure constraints are taken into account through appropriate Lagrange multipliers. Since the cut-joint, 6, is a revolute joint there are five Lagrange multipliers, three corresponding to the reaction forces and two to the reaction moments. Denoting the Lagrange multipliers $\lambda_1, \ldots, \lambda_5$, the reactions force, $\mathbf{f}_{65}$, and the reaction moment, $\mathbf{n}_{65}$, at the cut joint are given by

$$[\mathbf{f}_{65}]_6 = [\lambda_1 \quad \lambda_2 \quad \lambda_3]^T \quad \text{and} \quad [\mathbf{n}_{65}]_6 = [\lambda_5 \quad \lambda_6 \quad 0]^T \tag{41}$$

The subsystem II has 5 unknowns, namely, $\lambda_1, \ldots, \lambda_5$, which is equal to its dof. It implies that the five constrained equations [29] of the subsystem contain the five unknown multipliers. Therefore, the system of equations can be solved for the unknowns without considering subsystem I. This is the reason why the closed-loop is cut at joint 6 to make it open, Fig. 10. The reaction forces at the other joints, $1, \ldots, 5$, can then be calculated recursively using the methodology proposed in [22]. Knowing the vector of Lagrange multipliers, $\boldsymbol{\lambda}$, six unknowns of subsystem I, i.e., driving torque at joint 7, and three components of the reaction force, $\mathbf{f}_{76}$, and two components of the reaction moment, $\mathbf{n}_{76}$, can be solved easily. Note that the Z-component of $\mathbf{n}_{76}$ in frame $O_7X_7Y_7Z_7$ is nothing but the driving torque.

Having obtained the reactions at all the joints, the shaking force, and shaking moment about the mid-point of O_1O_7 on the fixed link are obtained using Eqs. (30) and (31) as

$$\mathbf{f}_{sh} = -(\mathbf{f}_{71} + \mathbf{f}_{76}) \tag{42}$$

$$\mathbf{n}_{sh} = -(\mathbf{n}_{71} + \mathbf{n}_{76} + 0.5\tilde{\mathbf{a}}_{17}\mathbf{f}_{76} - 0.5\tilde{\mathbf{a}}_{17}\mathbf{f}_{71}) \tag{43}$$

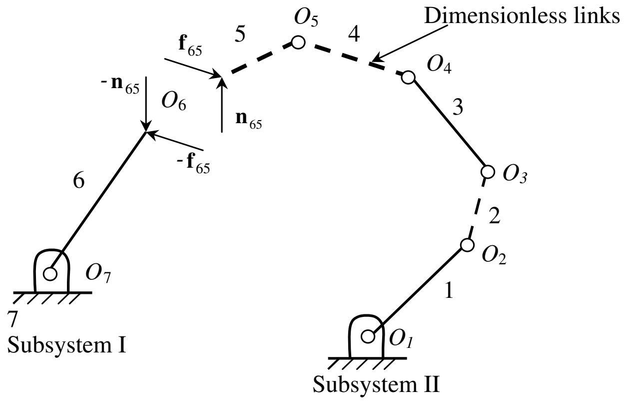

Fig. 10. Open system of the 7*R* mechanism.

Since only the links 1, 3, and 6, are having some dimensions with some masses, their equimomental systems of point-masses are brought in the optimization process. Hence, the 21-vector of the design variables is

$$\mathbf{x} \equiv [\mathbf{m}_1^{\mathrm{T}}, \mathbf{m}_3^{\mathrm{T}}, \mathbf{m}_6^{\mathrm{T}}]^{\mathrm{T}} \quad \text{for mass redistribution method}$$

$$\mathbf{x}^b \equiv [\mathbf{m}_1^{b^{\mathrm{T}}}, \mathbf{m}_3^{b^{\mathrm{T}}}, \mathbf{m}_6^{b^{\mathrm{T}}}]^{\mathrm{T}} \quad \text{for counterweight method}$$

Moreover, it is essential to position the mass centre of link 3 along O_3O_4 or X_4 so as to vanish the gyroscopic action on the mechanism [7], which is also taken into account as one of the constraints. The optimization problem is then posed as

$$\text{Minimize } z = w_1\tilde{f}_{\mathrm{sh}} + w_2\tilde{n}_{\mathrm{sh}} \tag{44a}$$

For mass redistribution

$$\text{Subject to} \quad \bar{m}_i^{\mathrm{o}} \leqslant \sum_{j=1}^{7} \bar{m}_{ij} \leqslant 5\bar{m}_i^{\mathrm{o}}; \quad r_i \leqslant a_i \tag{44b}$$

$$\sum_{j=1}^{7} m_{3j}[r_{3jy}]_4 = 0; \quad \text{and} \quad \sum_{j=1}^{7} m_{3j}[r_{3jz}]_4 = 0 \tag{44c}$$

For counterweight

$$\text{Subject to} \quad 0 \leqslant \sum_{j=1}^{7} m_{ij}^b \leqslant 3m_i^o; \quad 0 \leqslant r_i^b \leqslant a_i \tag{44d}$$

$$\sum_{j=1}^{7} (m_{3j}^{\mathrm{o}} + m_{3j}^b)[r_{3jx}]_4 = 0; \quad \sum_{j=1}^{7} (m_{3j}^{\mathrm{o}} + m_{3j}^b)[r_{3jy}]_4 = 0 \tag{44e}$$

for $i = 1, 3, 6$, where a_i is length of the ith link. Moreover $[\mathbf{r}_{3j}]_4 \equiv [\, r_{3jx} \quad r_{3jy} \quad r_{3jz}\,]^{\mathrm{T}}$ is the vector from O_4 to the point-mass, m_{3j}, expressed in frame $\mathscr{F}_4$, which is fixed to link 3. The two equality conditions of Eqs. (44c) and (44e), guarantee the location of the mass centre of link 3 along the axis, X_4, of the local frame, i.e., $\mathscr{F}_4$. Note here that the limits on the link masses and their mass centre locations are to be chosen by the designer. The optimization toolbox of MATLAB [25] is used to solve the optimization problem of Eq. (44). Using, "fmincon" function, which based on the Sequential Quadratic Programming (SQP) method [26], finds a minimum of the function z. The whole algorithm is coded in the MATLAB environment to determine the time dependent behaviour of the various relevant quantities. The 'fmincon' function of MATLAB is used which finds local minimum value of an objective function. Hence, the solution depends on the initial design vector taken to start the optimization process. The global solution is searched in the feasible space defined by the constraints, Eqs. (44b-c) or (44d-e), using different initial design vectors.

4.1. Numerical solution

Since the shaking force, shaking moment, bearing reactions, and parameters like the point masses and their distances from the origin of frame where they are attached have different units and magnitudes. In order to make them dimensionless, they are normalized as:
$a_{ij} =: a_{ij}/a_m$, normalized distance between joints i and j,
$r_i =: r_i/a_m$, normalized distance of the mass centre,
$m_i =: m_i/m_m^o$, normalized mass of the ith link,
$\mathbf{I}_i =: \mathbf{I}_i/(m_m^o a_m^2)$, normalized moment of inertia of the ith link,
$\bar{f} =: f/(m_m^o a_m \omega_{in}^2)$, normalized magnitude of force $\mathbf{f}$,
$\bar{n} =: n/(m_m^o a_m^2 \omega_{in}^2)$, normalized magnitude of moment/torque $\mathbf{n}$,

where a_m and m_m^o are the link length and original mass of reference link, m, about which the mechanism is normalized. Moreover, the scalar ω_{in} is the input angular speed.

Table 2 shows the DH parameters, mass and inertia of the normalized 7R mechanism. The mass centre location and the elements of the inertia tensor of each link are given in their local frames. The equimomental point masses and their locations are obtained using Eqs. (11)–(13) and (19) and are given in Table 3. The results of the optimization problem, Eq. (44), are obtained for three sets of weighting factors, namely, $(w_1, w_2) = (1.0, 0.0); (0.5, 0.5); (0.0, 1.0)$. Cases (1)–(3) corresponding to mass redistribution, whereas cases (4)–(6) represent counterweight balancing. Table 4 shows the optimum design vectors for all the cases. The geometry, mass and inertias for the normalized balanced mechanism corresponding to the cases are obtained using equimomental conditions and given in Table 5. Note that the conditions, Eqs. (44c) and (44e), are achieved and shown by bold-face number in Table 5. The results of case (1), which is shaking force balancing, is compared with the analytical conditions of Bagci [7]. The conditions are satisfied while the numerical values of Table 5 are used. Table 6 shows comparison to the RMS values of the normalized dynamic quantities occurring during motion cycle with those of the original mechanism, whereas Fig. 11 shows a comparison of the dynamic performances of the mechanism. It is obvious from Table 6 and Fig. 11 that a significant improvement in performances is achieved.

The following conclusions are accrued by comparison of the results given Tables 5 and 6.

- Total mass of the balanced mechanism is minimum when equal weights are applied to the shaking force and shaking moment, i.e., 3.199 for case (2) and 2.991 for case (5), for the mass redistribution and counterweight balancing, respectively.

Table 2
DH parameters, and mass and inertia properties for the normalized 7R mechanism

Link i	a_i	b_i	α_i	θ_i	m_i^o	r_{ix}^o	r_{iy}^o	r_{iz}^o	I_{ixx}^o	I_{iyy}^o	I_{izz}^o
1	1.00	0.55	90	θ_1	1.000	−0.50	0	0	0.0017	0.3477	0.3477
2	0	0	90	θ_2	0	0	0	0	0	0	0
3	1.10	0	90	θ_3	1.093	−0.55	0	0	0.0018	0.4579	0.4579
4	0	0	90	θ_4	0	0	0	0	0	0	0
5	0	0	90	θ_5	0	0	0	0	0	0	0
6	0.50	0	90	θ_6	0.536	−0.25	0	0	0.0009	0.0489	0.0489
7	1.30	0.40	150	θ_7	–	–	–	–	–	–	–

Total normalized mass of the mechanism, $\sum m_i^o = 2.629$. Normalized with respect to $a_1 = 0.1$ m, $m_1 = 0.084$ kg, $\omega_{in} = 1$ rad/sec.

Table 3
Equimomental point mass of the normalized original mechanism

Link i	Point-masses							Distances		
	m_{i1}^o	m_{i2}^o	m_{i3}^o	m_{i4}^o	m_{i5}^o	m_{i6}^o	m_{i7}^o	h_{ix}	h_{iy}	h_{iz}
1	0.2500	−0.2122	0.4622	0	0.4622	−0.2122	0.2500	0.5889	0.0292	0.0292
3	0.2732	−0.2324	0.5057	0	0.5057	−0.2324	0.2732	0.6466	0.0287	0.0287
6	0.1340	−0.1114	0.2454	0	0.2454	−0.1114	0.1340	0.3007	0.0290	0.0290

Table 4
Design vector for the normalized balanced mechanisms

Cases $[w_1, w_2]$	Design vector
(a) Mass redistribution method	
(1) [1.0,0.0]	[0.1212 −0.3017 0.6463 0.1430 0.6053 −0.3409 0.1606 0.4466 −0.1157 0.3313 −0.1157 0.3306 −0.1150 0.3309 0.5285 0.2829 −0.0096 −0.2548 −0.0976 0.1990 0.4442]^T
(2) [0.5,0.5]	[0.6521 −0.7163 −0.0964 0.4256 0.9352 0.2290 −0.4292 0.8384 1.1460 −0.7195 −0.7185 2.0459 −1.6194 0.1199 −0.6279 2.9295 −1.7194 1.1138 0.2628 −2.0301 1.1777]^T
(3) [0.0,1.0]	[−1.5240 −0.9882 0.9936 −0.4674 0.6876 0.4905 1.8079 0.3356 0.1637 0.1379 −0.0907 0.9190 −0.6174 0.2449 −0.1593 0.8132 −0.3126 0.2651 0.5540 −0.3610 0.6984]^T
(b) Counterweight method	
(4) [1.0,0.0]	[0.1716 −0.1736 0.1026 −0.0156 0.1825 −0.0923 0.0246 0.3383 0.0052 0.0120 −0.1606 0.0120 0.0052 0.1777 0.8146 0.6413 −0.0413 −0.1138 −0.3768 0.3060 0.3780]^T
(5) [0.5,0.5]	[−1.1917 −1.5375 0.4706 −0.0625 0.6477 0.9568 1.5094 0.1899 −0.1244 0.0609 −0.1264 0.0655 −0.1290 0.0635 0.2003 0.2840 −0.1789 −0.0969 −0.0285 0.2999 0.6255]^T
(6) [0.0,1.0]	[−2.8496 −3.6776 3.5322 3.8820 −6.2060 2.3213 3.9066 −0.8870 0.3400 0.1908 0.3576 −0.5485 1.0792 −0.5294 −1.9386 3.5731 0.4023 −1.4008 1.4840 0.6523 −1.1643]^T

Table 5
Geometry, mass and inertia of the normalized balanced mechanisms

Case	Link i	m_i^*	r_{ix}^*	r_{iy}^*	r_{iy}^*	I_{ixx}^*	I_{iyy}^*	I_{izz}^*	I_{ixy}^*	I_{iyz}^*	I_{izx}^*
(a) Mass redistribution method											
(1)	1	1.034	−1.000	0.005	0.001	0.0018	0.3594	0.3594	0.0032	0.0002	0.0031
	3	1.093	0.000	**0.000**	**0.000**	0.0018	0.4579	0.4579	−0.0043	−0.0002	−0.0043
	6	1.093	0.500	0.000	−0.009	0.0018	0.0997	0.0997	−0.0029	−0.0003	−0.0030
		3.220[a]									
(2)	1	1.000	−0.900	−0.014	0.108	0.0017	0.3477	0.3477	−0.0128	−0.0003	−0.0098
	3	1.093	−0.073	**0.000**	**0.000**	0.0018	0.4578	0.4578	−0.1293	0.0005	0.0760
	6	1.106	0.487	0.060	−0.096	0.0019	0.1010	0.1010	−0.0351	−0.0019	0.0860
		3.199									
(3)	1	1.000	−0.841	−0.145	−0.077	0.0017	0.3477	0.3477	0.0799	0.0012	0.0186
	3	1.093	−0.497	**0.000**	**0.000**	0.0018	0.4579	0.4579	−0.0324	0.0001	0.0256
	6	1.498	0.097	−0.006	−0.017	0.0025	0.1367	0.1367	−0.0080	−0.0001	0.0276
		3.591									
Case	Link i	m_i^{b*}	r_{ix}^{b*}	r_{iy}^{b*}	r_{iy}^{b*}	I_{ixx}^{b*}	I_{iyy}^{b*}	I_{izz}^{b*}	I_{ixy}^{b*}	I_{iyz}^{b*}	I_{izx}^{b*}
(b) Counterweight method											
(4)	1	0.200	−0.100	−0.004	0.043	0.0003	0.0695	0.0695	−0.0028	−0.0001	−0.0028
	3	0.390	1.100	**0.000**	**0.000**	0.0006	0.1633	0.1633	−0.0060	−0.0003	−0.0060
	6	1.608	0.500	0.018	−0.006	0.0027	0.1468	0.1468	−0.0048	−0.0005	−0.0048
		2.198[b]									
(5)	1	0.793	−0.980	−0.200	−0.003	0.0014	0.2756	0.2756	0.0852	0.0002	0.0055
	3	0.000	0.000	**0.000**	**0.000**	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	6	1.105	0.466	−0.018	−0.0093	0.0019	0.1009	0.1009	0.0017	−0.0003	0.0040
		1.898									
(6)	1	0.909	−0.977	0.028	−0.213	0.0015	0.3160	0.3160	0.4536	−0.0076	−0.0877
	3	0.003	0.695	**0.001**	**−0.001**	0.0000	0.0011	0.0011	0.0407	0.0017	−0.0142
	6	1.608	0.119	−0.006	−0.072	0.0027	0.1468	0.1468	−0.0404	0.0089	0.0294
		2.520									

[a] Total mass of the normalized mechanism, $\sum m_i^*$ for mass redistribution.
[b] For counterweight method $m_i^* = m_i^o + m_i^{b*}$, $\sum m_i^* = 3.198$; $\sum m_i^* = 2.991$; $\sum m_i^* = 3.056$ for cases (4)–(6), respectively.

- Total mass of the balanced mechanism in both the balancing methods are approximately same in the corresponding cases, e.g., 3.220 and 3.198 for cases (1) and (4), respectively.

Table 6
Comparison of the RMS values of the dynamic quantities

Case (w_1, w_2)	Input torque, $\tilde{\tau}$	Shaking force, $\tilde{f}_{sh}$	Shaking moment, $\tilde{n}_{sh}$*
Original	0.1279	0.7772	0.7458
(1) (1.0, 0.0)	0.1102 (−14)	0.0050 (−99)	0.2904 (−61)
(2) (0.5, 0.5)	**0.0940 (−27)**	**0.0875 (−89)**	**0.1119 (−92)**
(3) (0.0, 1.0)	0.0322 (−75)	0.4078 (−48)	0.1797 (−85)
(4) (1.0, 0.0)	0.0699 (−45)	0.3051 (−61)	0.3520 (−53)
(5) (0.5, 0.5)	**0.0639 (−50)**	**0.4614 (−41)**	**0.1785 (−76)**
(6) (0.0, 10)	0.0562 (−56)	0.6451 (−17)	0.2920 (−61)

The value in the parentheses denotes the round-off percentage increment over the corresponding value of the original mechanism.
* Shaking moment with respect to the mid-point of the fixed link 7.

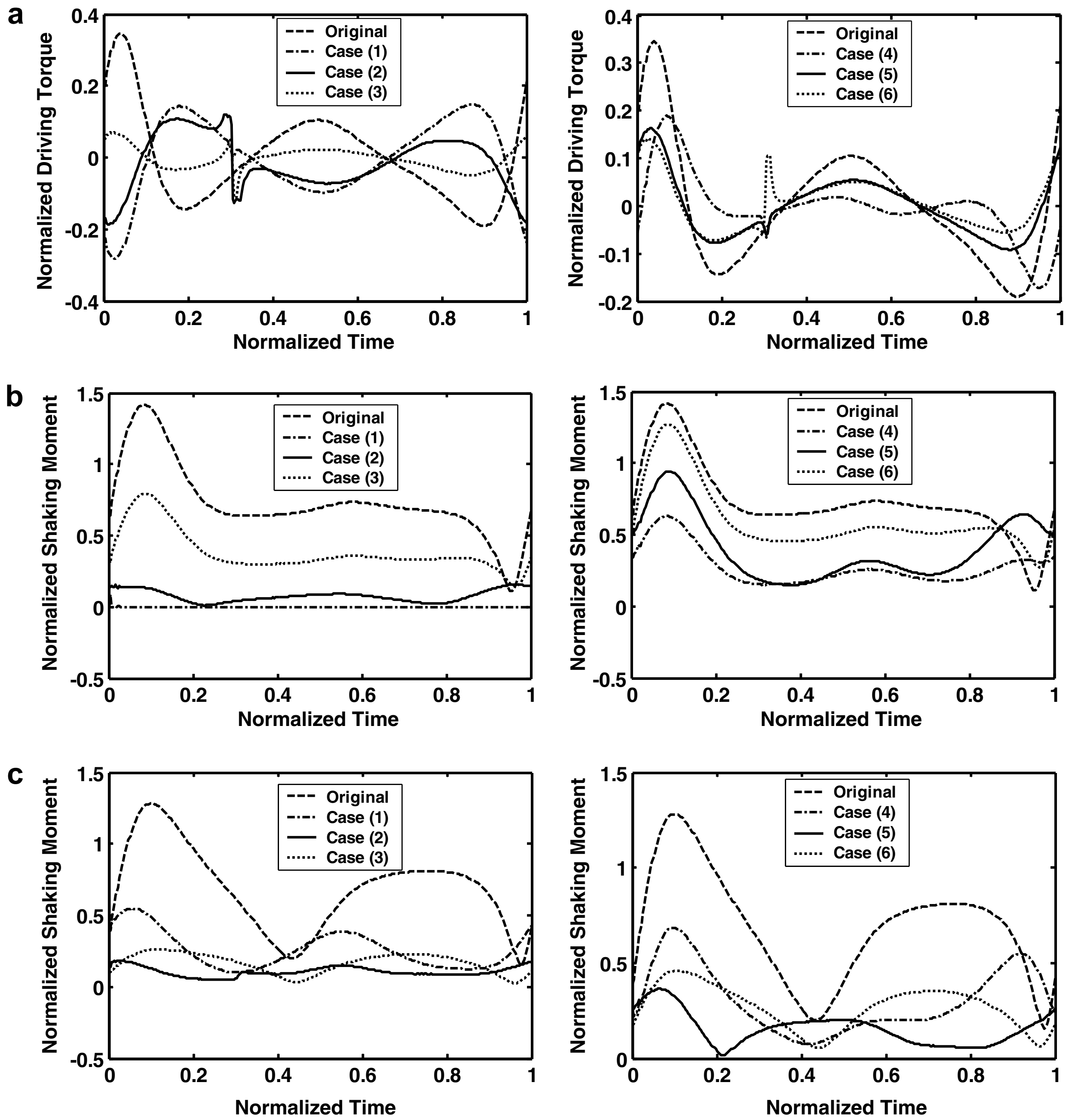

Fig. 11. Dynamic performance of the mechanism: (a) normalized driving torque; (b) normalized shaking force and (c) normalized shaking moment about the mid-point of the fixed link 7.

- Both the shaking force and shaking moment are reduced more in the mass redistribution balancing than counterweight balancing. With the mass redistribution, reduction in the RMS values of 89% and 92% is achieved for the shaking force and shaking moment, respectively, over that of the original mechanism whereas with counterweight balancing 41% and 76% reductions are seen in the RMS values of the shaking force and shaking moment, respectively.
- It is observed that when shaking moment with respect to O_1 is taken, the reactions at O_7 are minimum and vise-versa. Hence, to incorporate effect of both $\mathbf{f}_{71}$ and $\mathbf{f}_{76}$, the mid-point of O_1O_7 is chosen to determine the shaking moment.
- It is not necessary that the point-masses are positives. The condition is that a set of positive and negative point-masses represents a realizable link, i.e., the total mass and the moments of inertia about the axes passing through the centre of mass must be positive. Non-negativity of the total mass of each link is achieved by the constraint in the form of Eq. (35b). Non-zero moments of inertia are achieved by virtue of the positive link masses and their mass centre locations, as discussed after Eq. (36).

Based on the above observations, the mass redistribution method is effective. However, when this is not possible, counterweight balancing can be adopted. In order to realize the link shapes one may use small element superposing method [27], multi-layer link technique [28], and others, which is not discussed here further, as it is out of the scope of this paper.

5. Conclusions

This paper presents a generic optimization problem formulation for the balancing of shaking force and shaking moment of spatial mechanisms. In order to represent the inertial properties of a mechanism, the equimomental system of point-masses is introduced. The nonlinear equimomental conditions are converted into a set of conditions that are linear in point masses. This is achieved with the use of seven point-masses representing a rigid link. The constrained dynamic equations of motion are then obtained systematically in terms of the point-masses. To obtain these equations, appropriate joints of a closed-loop system are cut and the concept of the decoupled orthogonal complement matrices is used. The problem of mechanism balancing is then posed as an optimization problem to minimize the shaking force and shaking moment simultaneously due to inertia forces and moments. The results show that the shaking force and shaking moment are reduced more in the mass redistribution balancing in comparison to that in the counterweight balancing. This is quite obvious as the constraints are more stringent in the latter case. The effectiveness and the flexibility of the proposed method is illustrated using the spatial RSSR mechanism. A significant improvement in the performances is obtained in both the balancing methods compared to the original mechanism. In brief, the contributions of this paper are as follows:

(1) Introduction of seven point-mass system for a rigid link out of several choices so as to obtain the relevant parameters from a set of linear algebraic equations, namely, Eq. (15), instead of solving nonlinear equations, as required in the four point-masses and others.
(2) Solving the dynamics of a closed-loop system recursively which is known to be computationally efficient when the calculations are to be repeated several times, as in the optimization problem undertaken here.
(3) Formulation of balancing problem as optimization problem.
4) Providing practical solutions through link mass redistribution or using counterweights of the RSSR mechanism for its reduction in shaking force and shaking moment.

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